AFFINE ALGEBRAS, LANGLANDS DUALITY AND BETHE ANSATZ

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In memory of Claude Itzykson

1. INTRODUCTION.

By Langlands duality one usually understands a correspondence between automorphic representations of a reductive group G over the ring of adels of a field F, and homomorphisms from the Galois group $\mathrm{Gal}(\overline{F}/F)$ to the Langlands dual group G^L. It was originally introduced in the case when F is a number field or the field of rational functions on a curve over a finite field [1].

Recently A. Beilinson and V. Drinfeld [2] proposed a version of Langlands correspondence in the case when F is the field of rational functions on a curve X over $\mathbb{C}$. This geometric Langlands correspondence relates certain $\mathcal{D}$–modules on the moduli stack $\mathcal{M}_G(X)$ of principal G–bundles on X, and G^L–local systems on X (i.e. homomorphisms $\pi_1(X) \to G^L$). A. Beilinson and V. Drinfeld construct this correspondence by applying a localization functor to representations of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ of critical level $k = -h^\vee$, where $h^\vee$ is the dual Coxeter number.

The localization functor assigns a twisted $\mathcal{D}$–module on $\mathcal{M}_G(X)$ to an arbitrary $\widehat{\mathfrak{g}}$–module from a category $\mathcal{O}^0$. The fibers of this $\mathcal{D}$–module are analogous to spaces of conformal blocks from conformal field theory. In fact, the $\mathcal{D}$–module, which corresponds to the vacuum irreducible $\widehat{\mathfrak{g}}$–module of level $k \in \mathbb{Z}_+$, is the sheaf of sections of a vector bundle (with projectively flat connection), whose fiber is dual to the space of conformal blocks of the Wess-Zumino-Witten model. The analogy between conformal field theory and the theory of automorphic representations was underscored by E. Witten in [3]. It is at the critical level where this analogy can be made even more precise due to the richness of representation theory of $\widehat{\mathfrak{g}}$.

The peculiarity of the critical level is that a completion of the universal enveloping algebra of $\widehat{\mathfrak{g}}$ at this level, $U_{-h^\vee}(\widehat{\mathfrak{g}}) = U(\widehat{\mathfrak{g}})/(K + h^\vee)$, contains a large center $Z(\widehat{\mathfrak{g}})$. This center is isomorphic to the classical $\mathcal{W}$–algebra $\mathcal{W}(\mathfrak{g}^L)$ associated to the Lie algebra $\mathfrak{g}^L$, which is Langlands dual to $\mathfrak{g}$ [4]. Recall that $\mathcal{W}(\mathfrak{g}^L)$ consists of functionals on a certain Poisson manifold, which is obtained by the Drinfeld-Sokolov reduction from a hyperplane in the dual space to $\widehat{\mathfrak{g}^L}$ [5]. Elements of this Poisson manifold can be considered as connections of special kind on a G^L–bundle over a punctured disc called $\mathfrak{g}^L$–opers in [6]. For example, $\mathfrak{sl}_2$–opers are the same as projective connections. Thus,

$\mathfrak{g}^L$–opers define characters of $Z(\widehat{\mathfrak{g}})$, and one can obtain an infinite-dimensional family of $\widehat{\mathfrak{g}}$–modules of critical level by factoring Verma modules by these characters. The $\widehat{\mathfrak{g}}$–modules, which one obtains this way, play the role of unramified representations of the group G over a local field, while $\mathfrak{g}^L$–opers play the role of their local Langlands parameters.

By applying localization functor to these $\widehat{\mathfrak{g}}$–modules, one can associate a $\mathcal{D}$–module on $\mathcal{M}_G(X)$ to each regular $\mathfrak{g}^L$–oper on X. On the other hand, a regular $\mathfrak{g}^L$–oper defines a G^L–local system on X and hence a homomorphism $\pi_1(X) \to G^L$. This way one can establish, at least partially, the geometric Langlands correspondence [2].

When X is a rational curve, this correspondence can be analized rather explicitly. In this case we should consider the moduli space of G–bundles with parabolic structures at a finite number of marked points. One can then attach a $\mathcal{D}$–module on this moduli space to a $\mathfrak{g}^L$–oper with regular singularities at the marked points. For example, for $G = SL_2$ the $\mathcal{D}$–module thus obtained describes a system of differential equations: $H_i\Psi = \mu_i\Psi$, where μ_i's are the accessory parameters (residues at the marked points) of the corresponding projective connection, and H_i's are certain mutually commuting differential operators. These operators can be identified with the hamiltonians of the Gaudin model, which is a completely integrable quantum spin chain associated to $\mathfrak{g}$ [7]. The connection between Gaudin's model and affine algebras was studied in my joint work with B. Feigin and N. Reshetikhin [8]. There we gave a new interpretation of the Bethe ansatz.

Bethe ansatz is a method of diagonalization of commuting hamiltonians, which is widely used in various models of statistical mechanics. This method is one of the cornerstones of the powerful Quantum Inverse Scattering Method, which was intensively developed in the last twenty years, most notably, by the L.D. Faddeev School, cf. e.g. [9].

The idea of Bethe ansatz is to look for eigenvectors in a particular form. One can show that a vector written in such a form is an eigenvector if a certain system of algebraic equations, called the Bethe ansatz equations, is satisfied. The problem is to prove completeness of the Bethe ansatz, which means that all eigenvectors can be written in this special form.

For $G = SL_2$, I show in Sect. 5 below that the existence of an eigenvector of the Gaudin operators with given eigenvalues implies that the corresponding projective connection generates trivial monodromy representation of the fundamental group of $\mathbb{P}^1$ without the marked points. Remarkably, that is precisely equivalent to the Bethe ansatz equations. This observation enables us to prove completeness of Bethe ansatz in the Gaudin model associated to SL_2. Similar results can be obtained for other groups. For an alternative approach, cf. [46].

Let us now go back to the geometric Langlands correspondence. In [10] V. Drinfeld gave a beautiful construction of Langlands correspondence for the group GL_2 over the function field of a curve over $\mathbb{F}_q$, cf. also [11]. This construction is intrinsically geometric and can be carried out for a curve X over $\mathbb{C}$. Thus, one obtains two completely different realizations of Langlands correspondence for the group SL_2: one via the localization functor [2] and the other via the construction of [10]. An interesting question is to establish their equivalence.

In Sect. 6, I essentially do this in the case when the curve X is rational (these result is joint with B. Feigin). Namely, I show that the equivalence of the two constructions amounts to a separation of variables in the Gaudin system, which was introduced by E. Sklyanin [12] as an alternative to Bethe ansatz. It is possible that Sklyanin's separation of variables can be generalized to establish the equivalence of the two constructions of the geometric Langlands correspondence for curves of higher genus and groups of higher rank.

It is interesting that the Separation of Variables, possibly the most powerful method of solving quantum integrable models [13], appears as an ingredient of the geometric Langlands correspondence. Thus, Langlands philosophy unifies affine algebras, $\mathcal{D}$–modules on curves and on moduli spaces of bundles, and integrable systems. We believe that this is a part of an even richer picture, in which these relations are "deformed" in various directions. For instance, Sklyanin [14, 13] has found a separation of variables for the XXZ model, which is a q–deformation of a Gaudin model. This suggests that there should exist a "quantum Langlands correspondence" between certain systems of q-difference equations, in which the role of affine algebra is played by the corresponding quantum affine algebra. The latter also has a large center at the critical level [15, 16, 17, 18]. Moreover, elements of the spectrum of the center of a quantum affine algebra can be viewed as q–difference operators [18]. Hopefully, quantum affine algebras can bridge the gap between affine algebras and groups over a local non-archimedian field.

It is worth mentioning that besides geometric Langlands correspondence for complex curves, there are other intriguing new examples of Langlands duality [19]. We consider understanding this new duality pattern as one of the most challenging problems in contemporary mathematical physics.

2. Localization functor.

2.1. **Setup.** Let G be a connected simply-connected simple Lie group over $\mathbb{C}$, $\mathfrak{g}$ be its Lie algebra, and $\widehat{\mathfrak{g}}$ be the corresponding affine algebra – the extension of the (formal) loop algebra $L\mathfrak{g} = \mathfrak{g}((t))$ by one-dimensional center $\mathbb{C}K$ [20].

Fix a smooth projective curve X over $\mathbb{C}$, a point p on X and a formal coordinate near this point. Let $\mathcal{P}$ be a principal G–bundle over X with a trivialization on the formal disc around the point $p \in X$. Denote by $\mathfrak{g}_{\mathcal{P}} = \mathcal{P} \times_G \mathfrak{g}$ the vector bundle associated to the adjoint representation of G. Let $\mathfrak{g}_{\text{out}}^{\mathcal{P}}$ be the Lie algebra of sections of the bundle $\mathfrak{g}_{\mathcal{P}}$ over $X \backslash p$. There is a natural embedding $\mathfrak{g}_{\text{out}}^{\mathcal{P}} \to L\mathfrak{g}$ obtained by expanding sections in Laurent power series at p. It can be lifted to an embedding $\mathfrak{g}_{\text{out}}^{\mathcal{P}} \to \widehat{\mathfrak{g}}$.

Consider the category $\mathcal{O}^0$ of $\widehat{\mathfrak{g}}$–modules, on which the Lie subalgebra $\mathfrak{g}_{\mathrm{in}} = \mathfrak{g}[[t]] \subset \widehat{\mathfrak{g}}$ acts locally finitely. The modular functor assigns to a $\widehat{\mathfrak{g}}$–module M of an arbitrary level k from the category $\mathcal{O}^0$, the space of coinvariants $H(\mathfrak{g}, X, \mathcal{P}, M) = M/\mathfrak{g}_{\mathrm{out}}^{\mathcal{P}} M$, cf. [21]. The dual space to $H(\mathfrak{g}, X, \mathcal{P}, M)$ is called the *space of conformal blocks*. A conformal block is thus a linear functional f on M, which satisfies the *Ward identities*: $f(g \cdot x) = 0, \forall g \in \mathfrak{g}_{\mathrm{out}}^{\mathcal{P}}, \forall x \in M$.

The space $H(\mathfrak{g}, X, \mathcal{P}, M)$ depends on $\mathcal{P}$, and we want to study all of these spaces simultaneously. There is a standard way of doing that, due to Beilinson and Bernstein [22]. One constructs a *localization functor*, which assigns to a $\widehat{\mathfrak{g}}$–module M a twisted $\mathcal{D}$–module $\Delta(M)$ on the moduli space $\mathcal{M}_G(X)$ of G–bundles on X, such that the fiber of $\Delta(M)$ at $\mathcal{P}$ is isomorphic to $H(\mathfrak{g}, X, \mathcal{P}, M)$ (cf. e.g. [23, 24] for an introduction to the theory of $\mathcal{D}$–modules). Essentially, this just means realizing the Ward identities as differential equations on $\mathcal{M}_G(X)$, so that the space $H(\mathfrak{g}, X, \mathcal{P}, M)$ gets interpreted as the space of local solutions of those equations near $\mathcal{P} \in \mathcal{M}_G(X)$.[1]

To define this functor, observe that the moduli space of principal G–bundles with a trivialization on the formal disc around p is isomorphic to the homogeneous space $\mathcal{M}_G = LG/G_{\mathrm{out}}$, where $LG = G((t))$ is the Lie group of $L\mathfrak{g}$, and G_{out} is the Lie group of the Lie algebra $\mathfrak{g}_{\mathrm{out}}$ corresponding to the trivial bundle.

For any integer k one can define a line bundle ξ^k on $\mathcal{M}_G$ together with a homomorphism from $\widehat{\mathfrak{g}}$ to the Lie algebra of infinitesimal symmetries ξ^k, such that the central element K maps to the constant function k. This gives us a homomorphism from the algebra $U_k(\widehat{\mathfrak{g}}) = U(\widehat{\mathfrak{g}})/(K - k)$ to the algebra D'_k of global differential operators on ξ^k.

Let $\mathcal{D}'_k$ be the sheaf of differential operators on ξ^k. In fact, such a sheaf can be defined for any $k \in \mathbb{C}$, so that $\widehat{\mathfrak{g}}$ maps to its global sections, and K maps to k. For any $\widehat{\mathfrak{g}}$–module M of level k we can now define a left $\mathcal{D}'_k$–module on $\mathcal{M}_G$ as $\Delta'(M) = \mathcal{D}'_k \otimes_{U(\widehat{\mathfrak{g}})} M$. One can show as in [22], § 3, that the fiber of $\Delta'(M)$ at $\mathcal{P} \in \mathcal{M}_G$ is indeed isomorphic to $H(\mathfrak{g}, X, \mathcal{P}, M)$.

2.2. **Moduli space.** It is well-known that the set of isomorphism classes of principal G–bundles on X is isomorphic to the double quotient $G_{\mathrm{in}} \backslash LG/G_{\mathrm{out}}$, where $G_{\mathrm{in}} = G[[t]]$. However, this set is not the set of $\mathbb{C}$–points of an algebraic variety; the structure of algebraic variety can only be given to a subset of semi-stable G–bundles. One can cure this problem by considering the moduli stack $\mathcal{M}_G(X)$ of G–bundles on X. The precise definition of the algebraic stack $\mathcal{M}_G(X)$ can be found, e.g., in [25], § 3. This way we do not throw out any bundles: the set of isomorphism classes of $\mathbb{C}$–points of $\mathcal{M}_G(X)$ is by definition the set of isomorphism classes of all G–bundles on X. Moreover, $\mathcal{M}_G(X)$ is the stack theoretic quotient of the scheme (of infinite type) $\mathcal{M}_G = LG/G_{\mathrm{out}}$ by the algebraic group G_{in}.

The line bundle ξ^k on $\mathcal{M}_G$ is G_{in}–equivariant, and hence it descends to a line bundle η^k on $\mathcal{M}_G(X)$. Denote by $\mathcal{D}_k$ the sheaf of differential operators on η^k. Note that $\mathcal{D}_k$ can be defined for any $k \in \mathbb{C}$.

Since $M \in \mathcal{O}^0$, the action of the Lie subalgebra $\mathfrak{g}_{\mathrm{in}}$ on the module M is locally finite. Therefore we can integrate the action of $\mathfrak{g}_{\mathrm{in}}$ on M to an action of its Lie group G_{in}.

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[1]compare this with minimal models, where conformal blocks are solutions of differential equations on the moduli spaces of curves

The $\mathcal{D}'_k$–module $\Delta'(M)$ then becomes G_{in}–equivariant. Hence it is a pull-back of a $\mathcal{D}_k$-module $\Delta(M)$ on the moduli space $\mathcal{M}_G(X)$, cf. [26], § 4.5.

The spaces of coinvariants $H(\mathfrak{g}, X, \mathcal{P}, M)$, which are naturally isomorphic for different trivializations of $\mathcal{P}$ at p, are isomorphic to the fiber of $\Delta(M)$ at $\mathcal{P}$.

Examples. Let V_k be the vacuum Verma module over $\widehat{\mathfrak{g}}$ of level k, i.e. $V_k = U_k(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}_{\mathrm{in}})} \mathbb{C}$. Then $\Delta(V_k) = \mathcal{D}_k$. One obtains more interesting $\mathcal{D}$–modules by taking non-trivial quotients of V_k.

For instance, if k is a positive integer, then V_k contains a unique singular vector. Let L_k be the irreducible quotient of V_k. This is an integrable $\widehat{\mathfrak{g}}$–module. The spaces $H(\mathfrak{g}, X, \mathcal{P}, M)$ have the same dimension for different $\mathcal{P}$'s. Hence $\Delta(L_k)$ is the sheaf of sections of a vector bundle over $\mathcal{M}_G(X)$ with a projectively flat connection, whose fiber is $H(\mathfrak{g}, X, \mathcal{P}, M)$. $\qquad\square$

Remark. The localization functor first appeared in [27] and [28] in the following situation. Let B be the Borel subgroup of G and G/B be the flag manifold. To each integral weight λ one can associate a line bundle ξ_λ on G/B. There is a surjective homomorphism from $U(\mathfrak{g})$ to the algebra of differential operators on ξ_λ. This allows to define the localization functor from the category $\mathcal{O}$ of $\mathfrak{g}$–modules to the category of $\mathcal{D}_\lambda$–modules, where $\mathcal{D}_\lambda$ is the sheaf of differential operators on ξ_λ. It assigns to a $\mathfrak{g}$ module M, the $\mathcal{D}_\lambda$–module $\Delta(M) = \mathcal{D}_\lambda \otimes_{U(\mathfrak{g})} M$. This functor is exact if λ is dominant [27]. The adjoint functor assigns to a $\mathcal{D}_\lambda$–module, the $\mathfrak{g}$–module of its global sections. For example, if L_λ is the irreducible representation with an integral dominant weight λ, then $\Delta(L_\lambda)$ is just the sheaf of sections of ξ_λ. According to the Borel-Weil theorem, the space of global sections of ξ_λ is isomorphic to L_λ.

The space $\mathcal{M}_G$ can be thought of as an analogue of the flag manifold for the loop group, and hence the localization of $\widehat{\mathfrak{g}}$–modules described above is an analogue of the Beilinson-Bernstein localization. $\qquad\square$

We want to apply the localization functor Δ to $\widehat{\mathfrak{g}}$–modules of critical level $k = -h^\vee$. But first we review representation theory at this level.

3. Representations of critical level.

3.1. **Null-vectors and central elements.** Let us call a vector $x \in V_{-h^\vee}$ a *null-vector*, if $\mathfrak{g}_{\mathrm{in}} \cdot v = 0$. It follows from the Kac-Kazhdan formula for the determinant of the Shapovalov form [29] that V_k can contain null-vectors other than the highest weight vector v_k only if $k = -h^\vee$. From now on let us denote $V_{-h^\vee}$ by V and $v_{-h^\vee}$ by v. Each null-vector $x \in V$ defines an endomorphism e_x of V by the formula $e_x(Pv) = Px, \forall P \in U(\widehat{\mathfrak{g}})$, and each endomorphism e of V defines a null vector as $e \cdot v$. Hence the space $\mathfrak{z}(\widehat{\mathfrak{g}})$ of null-vectors in V is isomorphic to $\mathrm{End}_{\widehat{\mathfrak{g}}}(V)$.

Define the completion $\widetilde{U}(\widehat{\mathfrak{g}})$ of $U(\widehat{\mathfrak{g}})$ as the inverse limit of $U(\widehat{\mathfrak{g}})/U(\widehat{\mathfrak{g}})(\mathfrak{g} \otimes t^n \mathbb{C}[t]), n > 0$. The action of $\widetilde{U}(\widehat{\mathfrak{g}})$ is well-defined on all modules from the category $\mathcal{O}^0$. Using the structure of vertex operator algebra on V [30, 4], we can attach to each vector $P \in V$ a power series $Y(P, z) = \sum_{m \in \mathbb{Z}} P_m \, z^m$ (for precise definition, cf. [31]). The coefficients of these power series are elements of $\widetilde{U}_{-h^\vee}(\widehat{\mathfrak{g}}) = \widetilde{U}(\widehat{\mathfrak{g}})/(K + h^\vee)$, and altogether they span a Lie subalgebra $U_{-h^\vee}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$ of $\widetilde{U}_{-h^\vee}(\widehat{\mathfrak{g}})$, which is called the local completion of

$U_{-h^\vee}(\widehat{\mathfrak{g}})$ [4]. For example, for each $A \in \mathfrak{g}$, we have: $Y((A \otimes t^{-1})v, z) \equiv A(z) = \sum_{n \in \mathbb{Z}} (A \otimes t^n) z^{-n-1}$. This shows that $\widehat{\mathfrak{g}} \subset U_{-h^\vee}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$.

Let $Z(\widehat{\mathfrak{g}})$ be the center of $U_{-h^\vee}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$. Since $\widehat{\mathfrak{g}} \subset U_{-h^\vee}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$, elements of $Z(\widehat{\mathfrak{g}})$ are central in $\widetilde{U}_{-h^\vee}(\widehat{\mathfrak{g}})$. It is easy to show that if $x \in \mathfrak{z}(\widehat{\mathfrak{g}})$, then all coefficients of $Y(x, z)$ lie in $Z(\widehat{\mathfrak{g}})$. Furthermore, all elements of $Z(\widehat{\mathfrak{g}})$ can be obtained this way, and the natural map $Z(\widehat{\mathfrak{g}}) \to \mathfrak{z}(\widehat{\mathfrak{g}})$ defined by the formula $c \to c \cdot v$ is surjective [4]. In particular, we see that the algebra $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathrm{End}_{\widehat{\mathfrak{g}}}(V)$ is commutative.

Example. Let $\{J_a\}, a = 1, \ldots, \dim \mathfrak{g}$, be a basis of $\mathfrak{g}$ orthonormal with respect to an invariant bilinear form. The vector $S = \frac{1}{2} \sum_a (J_a \otimes t^{-1})^2 v$ lies in $\mathfrak{z}(\widehat{\mathfrak{g}})$. The coefficients S_n of the power series

$$(3.1) \qquad Y(S, z) \equiv \sum_{n \in \mathbb{Z}} S_n z^{-n-2} = \frac{1}{2} \sum_a : J_a(z)^2 :,$$

where the columns stand for the normal ordering, lie in $Z(\widehat{\mathfrak{g}})$. They are called the *Sugawara operators.* $\square$

3.2. **Symbols of null-vectors.** Recall that the center of $U(\mathfrak{g})$ is a free polynomial algebra with generators C^i of orders $d_i + 1, i = 1, \ldots, l$, where $d_1, \ldots, d_l$ are the exponents of $\mathfrak{g}$. In particular, $C^1 = \frac{1}{2} \sum_a (J_a)^2$. It is natural to try to generalize the formula for S by taking other generators $C^i, i > 1$, and replacing each J_a by $J_a \otimes t^{-1}$. Unfortunately, this approach does not produce elements of $\mathfrak{z}(\widehat{\mathfrak{g}})$ (and hence $Z(\widehat{\mathfrak{g}})$) in general. Some constructions have been given for the affine algebras of types $A_n^{(1)}, B_n^{(1)}$ and $C_n^{(1)}$ [32], but explicit formulas for elements of $\mathfrak{z}(\widehat{\mathfrak{g}})$ are not known in general.

The standard filtration on $U(\widehat{\mathfrak{g}})$ induces a natural filtration on the module V, such that $\mathrm{gr}\, V$ is isomorphic to $S^*(L\mathfrak{g}/\mathfrak{g}_{\mathrm{in}})$. Moreover, $\mathrm{gr}\, \mathfrak{z}(\widehat{\mathfrak{g}})$ is isomorphic to the space of $\mathfrak{g}_{\mathrm{in}}$–invariants of $S^*(L\mathfrak{g}/\mathfrak{g}_{\mathrm{in}})$. Similarly, $\mathrm{gr}\, U(\mathfrak{g}) \simeq S^*(\mathfrak{g})$, and it is known that the center of $U(\mathfrak{g})$ is isomorphic to the $\mathfrak{g}$–invariant part of $S^*(\mathfrak{g})$. We can apply our naive approach to the symbols $\bar{C}^i$'s of the central elements C^i's of $U(\mathfrak{g})$, i.e. replace J_a by $J_a \otimes t^{-1}$ in $\bar{C}^i$ and consider it as an element $\bar{S}^i$ of $S^*(L\mathfrak{g}/\mathfrak{g}_{\mathrm{in}})$. It is easy to check that each $\bar{S}^i$ is $\mathfrak{g}_{\mathrm{in}}$–invariant. One can therefore ask whether each $\bar{S}^i$ can be lifted to some $S^i \in \mathfrak{z}(\widehat{\mathfrak{g}})$.

The results stated in the next section give us the affirmative answer to this question for an arbitrary affine Lie algebra.

3.3. **The structure of the center.** Recall that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a commutative algebra isomorphic to $\mathrm{End}_{\widehat{\mathfrak{g}}}(V)$. The Lie algebra of vector fields on the punctured disc acts on $\widehat{\mathfrak{g}}$ according to the formulas: $[t^n \partial_t, A \otimes t^m] = m A \otimes t^{n+m-1}$. This action induces an action of the Lie algebra of regular vector fields on the disc on V if we put $t^n \partial_t \cdot v = 0, n \geq 0$. In particular, the vector field $-t \partial_t$ provides a $\mathbb{Z}$–grading on V, such that $\deg A \otimes t^n = -n$ and $\deg v = 0$. One can show that $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset V$ is preserved by this action.

Theorem 1 ([4]). *There exist $S^1, \ldots, S^l \in \mathfrak{z}(\widehat{\mathfrak{g}})$, such that $\deg S^i = d_i + 1$, and $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathbb{C}[\partial_t^n S^i]_{i=1,\ldots,l; n \geq 0}$. In particular, $S^1 = \frac{1}{2} \sum_a (J_a \otimes t^{-1})^2 v$.*

The theorem implies that the center $Z(\widehat{\mathfrak{g}})$ of $U_{-h^\vee}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$ is generated in the appropriate sense by the Fourier coefficients of the power series $S^i(z) \equiv Y(S^i, z), i = 1, \ldots, l,$

corresponding to S^i's. We will give a more precise description of $Z(\widehat{\mathfrak{g}})$ using the Poisson structure, which is defined as follows.

For any $x_1, x_2 \in Z(\widehat{\mathfrak{g}}) \subset \widetilde{U}_{-h^\vee}(\widehat{\mathfrak{g}})$, let $\widetilde{x}_1, \widetilde{x}_2$ be their liftings to $\widetilde{U}(\widehat{\mathfrak{g}})$. Then $[\widetilde{x}_1, \widetilde{x}_2] = (K + h^\vee)\widetilde{y} + (K + h^\vee)^2(\dots)$, for some $\widetilde{y} \in \widetilde{U}(\widehat{\mathfrak{g}})$. Put $\{x_1, x_2\} = y \equiv \widetilde{y} \bmod (K + h^\vee)$. One can check that $y \in Z(\widehat{\mathfrak{g}})$, and that this operation is a Poisson bracket on $Z(\widehat{\mathfrak{g}})$.[2] The Lie algebra of vector fields on the punctured disc acts on $U_{-h^\vee}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$, and this action preserves $Z(\widehat{\mathfrak{g}})$. Moreover, the induced action on $Z(\widehat{\mathfrak{g}})$ is hamiltonian: the vector fields $-t^{n+1}\partial_t$ act as $\{S_n, \cdot\}$, where S_n are the Sugawara operators (3.1).

Recall that for any simple Lie algebra $\mathfrak{g}$ one can define its Langlands dual $\mathfrak{g}^L$ as the Lie algebra whose Cartan matrix is the transpose of the Cartan matrix of $\mathfrak{g}$. The following description of the center $Z(\widehat{\mathfrak{g}})$ of $U_{-h^\vee}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$ was conjectured by Drinfeld.

Theorem 2 ([4]). $Z(\widehat{\mathfrak{g}})$ *is isomorphic to the classical* $\mathcal{W}$*–algebra* $\mathcal{W}(\mathfrak{g}^L)$ *associated to* $\mathfrak{g}^L$, *as a Poisson algebra. This isomorphism is equivariant with respect to the action of the Lie algebra of vector fields on the punctured disc.*

Classical $\mathcal{W}$–algebras are defined via the Drinfeld-Sokolov hamiltonian reduction [5]. Let us recall this construction.

3.4. **Hamiltonian reduction.** Let $C(\mathfrak{g})$ be the space of connections on the trivial G–bundle over the punctured disc (more precisely, on $\mathrm{Spec}\,\mathbb{C}((t))$). Such a connection is a first order differential operator $\partial_t + A(t)$, where $A(t) \in \mathfrak{g} \otimes \Omega^1 = \mathfrak{g} \otimes \mathbb{C}((t))dt$.

Let $(\cdot, \cdot)$ be the standard invariant bilinear form on $\mathfrak{g}$ normalized so that the square of the maximal root is equal to 2 [20]. Denote by $r^\vee$ the maximal number of edges connecting two vertices of the Dynkin diagram of $\mathfrak{g}$. Using the form $(\cdot, \cdot)$ we can identify the space $C(\mathfrak{g})$ with the hyperplane in $\widehat{\mathfrak{g}}^*$, which consists of linear functionals on $\widehat{\mathfrak{g}}$ taking the value $1/r^\vee$ on K. This hyperplane is equipped with a canonical Poisson structure, which is the restriction of the Kirillov-Kostant structure on $\widehat{\mathfrak{g}}^*$. The space of local functionals on $C(\mathfrak{g})$ is then a Poisson algebra.

We have a natural projection $\imath : C(\mathfrak{g}) \to (L\mathfrak{n}_+)^*$, which can be interpreted as the moment map with respect to the coadjoint action of the Lie group LN_+. We now perform the hamiltonian reduction with respect to the one-point orbit $\chi \in (L\mathfrak{n}_+)^*$, such that $\chi(e_\alpha \otimes t^n) = -1$, if α is a simple root and $n = -1$, and $\chi(e_\alpha \otimes t^n) = 0$ otherwise. This means that we take the inverse image of χ and take its quotient by the action of LN_+. It turns out that this action is free [5], and the quotient is an infinite-dimensional affine space $\mathcal{C}(\mathfrak{g})$ which is isomorphic (non-canonically) to $\mathbb{C}((t))^l$.

Remark. Following Drinfeld, consider the inverse image $\mathcal{C}'(\mathfrak{g})$ of the set of all $\chi' \in (L\mathfrak{n}_+)^*$, such that $\chi'(e_\alpha \otimes t^n) \neq 0$ for some $n \in \mathbb{Z}$, if α is simple, and $\chi'(e_\alpha \otimes t^n) = 0$ otherwise. Then the loop group LB_+ of the Borel subgroup $B_+ \subset G$ acts on $\mathcal{C}'(\mathfrak{g})$, and the quotient $\mathcal{C}'(\mathfrak{g})/LB_+$ is isomorphic to $\mathcal{C}(\mathfrak{g})$. After this identification, we obtain a natural action of vector fields on the punctured disc on $\mathcal{C}(\mathfrak{g})$, which is mentioned in Theorem 2. One can show that with respect to this action $\mathcal{C}(\mathfrak{g})$ is isomorphic to the product of the space of projective connections (cf. below) and $\oplus_{i=2}^l \Omega^{d_i+1}$. $\square$

The classical $\mathcal{W}$–algebra $\mathcal{W}(\mathfrak{g})$ is by definition the space of local functionals on $\mathcal{C}(\mathfrak{g})$.

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[2]This was observed by V. Drinfeld following T. Hayashi's work [32].

Example. Here is a more explicit description of $\mathcal{W}(\mathfrak{sl}_n)$. In this case each LN_+–orbit in $\imath^{-1}(\chi)$ contains a unique element of the form

(3.2)
$$\partial_t - \begin{pmatrix} 0 & q_1(t) & q_2(t) & \ldots & q_{n-2}(t) & q_{n-1}(t) \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \hdotsfor{6} \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

which can be viewed as the nth order differential operator

(3.3)
$$\partial_t^n - q_1(t)\partial_t^{n-2} - \ldots - q_{n-1}(t).$$

The isomorphism of Theorem 2 identifies the local functionals $q_i(m) = \int q_i(t)\, t^{m+i} dt$ with central elements $S_m^i = \int S^i(z) z^{m+i} dz \in Z(\widehat{\mathfrak{sl}}_n)$. $\qquad\qquad\square$

A similar description of the spaces $\mathcal{C}(\mathfrak{g})$ for classical Lie algebras was given in [5].

3.5. **Opers.** Following [6], we will call elements of $\mathcal{C}(\mathfrak{g})$, $\mathfrak{g}$–*opers* on the punctured disc. Similarly, one can define the space $\mathcal{C}_+(\mathfrak{g})$ of *regular* $\mathfrak{g}$–opers on the disc. Beilinson and Drinfeld have generalized this notion to the case of an arbitrary curve X [6]. Let G^{ad} be the group of inner automorphisms of $\mathfrak{g}$. A $\mathfrak{g}$–oper on X is a G^{ad}–bundle over X with a connection and an additional structure, such that locally it can be considered as an element of $\mathcal{C}_+(\mathfrak{g})$.

For example, an $\mathfrak{sl}_n$–oper on a curve X is a rank n (holomorphic) vector bundle E over X defined up to tensoring with a line bundle, equipped with a full flag of subbundles $0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$ and a connection $\nabla : E \to E \otimes \Omega^1$, such that $\nabla \cdot E_i \subset E_{i+1} \otimes \Omega^1$ and the induced maps $E_i/E_{i-1} \to E_{i+1}/E_i \otimes \Omega^1$ are isomorphisms for $i = 1, \ldots, n-1$. Thus, $E_i/E_{i-1} \simeq E_1 \otimes \Omega^{i-1}$, and after tensoring with the appropriate line bundle we can set $E_i = \Omega^{i-(n+1)/2}$. As a PGL_n–bundle, E is uniquely defined by these properties, in particular, it does not depend on the choice of theta-characteristic.

Over the punctured disc, the bundle E can be trivialized, and the connection ∇ can be brought to the form (3.2). Hence we obtain an equivalent definition: an $\mathfrak{sl}_n$–oper is an nth order differential operator acting from $\Omega^{(-n+1)/2}$ to $\Omega^{(n+1)/2}$, such that locally it has the form (3.3), i.e. its principal symbol is equal to 1, and the sub-principal symbol vanishes.

3.6. **$\widehat{\mathfrak{g}}$–modules associated to opers.** Let $M_{\chi,k}$ be the Verma module over $\widehat{\mathfrak{g}}$. Recall that $M_{\chi,k} = U_k(\widehat{\mathfrak{g}}) \otimes_{U(\widetilde{\mathfrak{b}}_+)} \mathbb{C}_\chi$, where $\widetilde{\mathfrak{b}}_+ = (\mathfrak{b}_+ \otimes 1) \oplus (\mathfrak{g} \otimes t\mathbb{C}[[t]])$ is the standard Borel subalgebra of $\widehat{\mathfrak{g}}$, and $\chi \in \mathfrak{h}^*$. Denote by $v_{\chi,k}$ the highest weight vector of $M_{\chi,k}$. Let $\rho \in \mathcal{C}(\mathfrak{g}^L)$ be a $\mathfrak{g}^L$–oper on the punctured disc. By Theorem 2, ρ defines a central character, i.e. a homomorphism $\widetilde{\rho} : Z(\widehat{\mathfrak{g}}) \to \mathbb{C}$. We define a $\widehat{\mathfrak{g}}$–module $M_\chi^\rho = M_{\chi,-h^\vee} / \mathrm{Ker}\,\widetilde{\rho}$.

Example. The space $\mathcal{C}(\mathfrak{sl}_2)$ consists of projective connections on the punctured disc, i.e. operators of the form $\partial_t^2 - q(t)$ acting from $\Omega^{-1/2}$ to $\Omega^{3/2}$, where $q(t) = \sum_{n \in \mathbb{Z}} q_n t^{-n-2}$. This space is an Ω^2–torsor.

The module $M_\chi^{q(t)}$ is by definition the quotient of $M_{\chi,-2}$ by the submodule generated by the vectors $[S_m - q_m]v_{\chi,-2}, m \in \mathbb{Z}$. But we know that $S_m v_{\chi,-2} = 0$ for $m > 0$ and

$S_0 v_{\chi,-2} = \frac{1}{2}\sum_a [J_a \otimes 1]^2 v_{\chi,-2} = \frac{1}{4}\chi(\chi+2)v_{\chi,-2}$. Therefore $M_{\chi,-2}^{q(t)} \neq 0$ if and only if $q_m = 0, m > 0$, and $q_0 = \frac{1}{4}\chi(\chi+2)$. This means that the projective connection $\partial_t^2 - q(t)$ has regular singularity at the origin, and $q(t) = \frac{1}{4}\chi(\chi+2)/t^2 + \dots$. $\qquad\square$

Likewise, when $\mathfrak{g} = \mathfrak{sl}_n$, the quotient of the Verma module $M_{\chi,-h^\vee}$ defined by the differential operator (3.3) is non-zero if and only if this operator has regular singularity at the origin and the most singular terms are equal to the values of S_0^i on the highest weight vector of $M_{\chi,-h^\vee}$. In general, we also have an infinite-dimensional family of $\widehat{\mathfrak{g}}$–modules of critical level parametrized by $\mathfrak{g}^L$–opers with regular singularities (in the sense of [6]).

Another family of $\widehat{\mathfrak{g}}$–modules can be obtained from the module V. One can show that $\operatorname{Spec} \operatorname{End}_{\widehat{\mathfrak{g}}}(V) \simeq \mathcal{C}_+(\mathfrak{g}^L)$, the space of regular $\mathfrak{g}^L$–opers on the formal disc. Therefore each $\rho \in \mathcal{C}_+(\mathfrak{g}^L)$ defines a character $\bar\rho$ of $\operatorname{End}_{\widehat{\mathfrak{g}}}(V)$. We set $V^\rho = V/\operatorname{Ker} \bar\rho$.

3.7. Wakimoto modules.
There is another very important family of $\widehat{\mathfrak{g}}$–modules of critical level – the Wakimoto modules.

To motivate the definition of Wakimoto modules, recall that in the study of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ an important role is played by the Harish-Chandra homomorphism $Z(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*]$. The image of $z \in Z$ is a polynomial on $\mathfrak{h}^*$, whose value at $\lambda \in \mathfrak{h}^*$ is equal to the value of z on the Verma module M_λ.

In order to generalize this construction to the affine case with $L\mathfrak{h}^* = \mathfrak{h}^* \otimes \Omega^1$ instead of $\mathfrak{h}^*$, we need $\widehat{\mathfrak{g}}$–modules, which depend on elements of $L\mathfrak{h}^*$. The Verma modules can not be used as they depend only on elements of $\mathfrak{h}^*$.

Roughly, we would like take a one-dimensional representation of $L\mathfrak{h}$, extend it trivially to a $L\mathfrak{n}_-$ and take the induced $L\mathfrak{g}$–module. But the Lie algebra $L\mathfrak{n}_+$ would act freely on the induced module. Hence the action of the completion $\widetilde{U}(\widehat{\mathfrak{g}})$ on such a module is not well-defined, and we can not use them to construct an analogue of the Harish-Chandra homomorphism.

However, we can define a $\widehat{\mathfrak{g}}$–module, which is partially induced and partially coinduced [34], so that the Lie algebra $\mathfrak{n}_+ \otimes t^{-1}\mathbb{C}[t^{-1}]$ acts freely and the Lie algebra $\mathfrak{n}_+ \otimes \mathbb{C}[t]$ acts cofreely. On these modules the action of $\widetilde{U}(\widehat{\mathfrak{g}})$ well-defined. But the definition of these modules requires regularization (normal ordering) and this leads to a shift in the level – it becomes $-h^\vee$. These modules are called Wakimoto modules, cf. [36] for a review.

Example. If $\mathfrak{g} = \mathfrak{sl}_2$, then the Wakimoto module of critical level has the following realization [35]. Consider the Heisenberg algebra Γ with generators $a_n, a_n^*, n \in \mathbb{Z}$, and relations $[a_n, a_m^*] = \delta_{n,-m}$. Let M be the Fock representation of this algebra generated by a vector $\mathbf{v}$, such that $a_n\mathbf{v} = 0, n \geq 0; a_n^*\mathbf{v} = 0, n > 0$. Put $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, a^*(z) = \sum_{n\in\mathbb{Z}} a_n^* z^{-n}$. Let $\{e, h, f\}$ be the standard basis of $\mathfrak{sl}_2$. For any formal Laurent power series $\chi(z) = \sum_{n\in\mathbb{Z}} \chi_n z^{-n-1}$, define an action of $\widehat{\mathfrak{sl}}_2$ on M by the formulas:

$$e(z) = a(z), \qquad h(z) = -2 : a(z)a^*(z) : + \chi(z),$$
$$f(z) = - : a(z)a^*(z)a^*(z) : -2\partial_z a^*(z) + \chi(z)a^*(z).$$

The corresponding $\widehat{\mathfrak{sl}}_2$–module is the Wakimoto module; we denote it by $W_{\chi(z)}$. $\qquad\square$

Because of normal ordering, if we perform a change of coordinates, $\chi(t)$ will transform not as a one-form, but as a connection $\partial_t + \frac{1}{2}\chi(t) : \Omega^{-1/2} \to \Omega^{1/2}$ over the punctured disc. Similarly, the Wakimoto modules over $\widehat{\mathfrak{g}}$ are parametrized by connections on a certain H^L–bundle, where H^L is the dual group to the Cartan subgroup H of G. This space is an $\mathfrak{h}^* \otimes \Omega^1$–torsor. If we choose a coordinate t on the disc, we can identify it with $\mathfrak{h}^* \otimes \mathbb{C}((t))$.

Now let us return to the center $Z(\widehat{\mathfrak{sl}}_2)$. As we know, it is generated by elements S_n given by formula (3.1). To describe the action of $Z(\widehat{\mathfrak{g}})$ on $W_{\chi(t)}$, it is therefore sufficient to describe the action of the operators S_n. The operator S_n acts on $W_{\chi(t)}$ by multiplication by some $q_n \in \mathbb{C}$. Set $q(t) = \sum_{n \in \mathbb{Z}} q_n z^{-n-2}$. The projective connection corresponding to the action of $Z(\widehat{\mathfrak{sl}}_2)$ on $W_{\chi(t)}$ is then $\partial_t^2 - q(t)$. One can show by direct computation that the relation between this projective connection and the connection $\partial_t + \frac{1}{2}\chi(t)$ has the form:

$$(3.4) \qquad \partial_t^2 - q(t) = (\partial_t - \frac{1}{2}\chi(t))(\partial_t + \frac{1}{2}\chi(t)),$$

which means that

$$(3.5) \qquad q(t) = \frac{1}{4}\chi(t)^2 - \frac{1}{2}\partial_t\chi(t).$$

3.8. **Miura transformation.** Formula (3.5) is the transformation law from $\partial_t + \frac{1}{2}\chi(t)$ to $\partial_t^2 - q(t)$, which is called the *Miura transformation*. An equivalent way to define it is by saying that the Miura transformation of the connection

$$\partial_t - \begin{pmatrix} \chi(t)/2 & 0 \\ 1 & -\chi(t)/2 \end{pmatrix}$$

on the trivial rank two vector bundle on the punctured disc is the unique gauge equivalent connection of the form

$$\partial_t - \begin{pmatrix} 0 & q(t) \\ 1 & 0 \end{pmatrix}.$$

One can also say this in a coordinate independent way, suitable for an arbitrary curve X. Recall that an $\mathfrak{sl}_2$–oper on X is a rank 2 bundle E which is a non-trivial extension $0 \to \Omega^{1/2} \to E \to \Omega^{-1/2} \to 0$, and a connection ∇, which defines an isomorphism between $\Omega^{1/2} \subset E$ and $(E/\Omega^{1/2}) \otimes \Omega^1$. Suppose there is another subbundle of rank one, $E' \subset E$, which is preserved by the connection ∇, i.e. $\nabla \cdot E' \subset E' \otimes \Omega^1$. The Miura transformation is by definition the transformation from the local system (E', ∇') to the oper (E, ∇).[3]

In general, the action of the center $Z(\widehat{\mathfrak{g}})$ on the Wakimoto modules is given by the generalized Miura transformation introduced in [5].[4] This fact follows directly from our construction of the isomorphism between the center $Z(\widehat{\mathfrak{g}})$ and the $\mathcal{W}$–algebra $\mathcal{W}(\mathfrak{g}^L)$ [4], cf. also [8]. Thus, the affine analogue of the Harish-Chandra homomorphism is the Miura transformation.

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[3]The subbundle E' does not exist if X is a compact curve of genus $g > 1$. However, an invariant subbundle E' may exist on an affine curve, e.g. on a punctured disc.

[4]this is a hamiltonian map relating the Poisson structures of the generalized KdV and mKdV equations

In the case of $\mathfrak{sl}_n$ the Miura transformation can be described as follows. Suppose that we have an oper (E, ∇) with another full flag of subbundles $E_1' \subset E_2' \subset \ldots E_n'$, which is preserved by the connection, i.e. $\nabla \cdot E_i' \subset E_i' \otimes \Omega^1$ for all $i = 1, \ldots, n$. Such a flag defines a connection ∇' on the direct sum $E' = \oplus_{i=1}^n E_i'/E_{i-1}'$. The Miura transformation is the transformation from the local system (E', ∇') to the oper (E, ∇).

Locally, in the basis induced by the flag $E_\bullet$ the connection ∇ can be represented by formula (3.2), whereas in the basis induced by the flag $E_\bullet'$, ∇ can be represented as

$$(3.6) \qquad \partial_t - \begin{pmatrix} \chi_1(t) & 0 & \ldots & 0 & 0 \\ 1 & \chi_2(t) & \ldots & 0 & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & 0 & \ldots & \chi_{n-1}(t) & 0 \\ 0 & 0 & \ldots & 1 & \chi_n(t) \end{pmatrix}.$$

Thus, locally the Miura transformation is just the transformation from the connection (3.6) to the unique gauge equivalent connection of the form (3.2), cf. [5]. This amounts to the following splitting of the differential operator (3.3) into the product of the first order linear operators:

$$\partial_t^n - q_1(t)\partial_t^{n-2} - \ldots - q_{n-1}(t) = (\partial_t - \chi_1(t))\ldots(\partial_t - \chi_n(t)).$$

3.9. $\widehat{\mathfrak{g}}$–modules and connections.

As we have seen, $\widehat{\mathfrak{g}}$–modules of critical level are parametrized by geometric data, namely, local systems on the formal punctured disc. Moreover, many properties of these local systems have representation theoretic meaning. Here is an example, which we will need in § 5.

Let us look at the Miura transformation (3.5) as the Riccati equation on $\chi(t)$ with fixed $q(t)$. Assume that the projective connection $\partial_t^2 - q(t)$ has at most regular singularity at the origin, i.e. that $q(t) = \sum_{n \leq 0} q_n t^{-n-2}$. In that case the Riccati equation (3.5) can be considered as a system of algebraic equations on the Fourier coefficients of $\chi(t) = \sum_{m \leq 0} \chi_m t^{-m-1}$:

$$(3.7) \qquad \frac{1}{4} \sum_{i+j=n} \chi_i \chi_j + \frac{n+1}{2} \chi_n = q_n, \qquad n \leq 0.$$

Note that the first equation is $q_0 = \frac{1}{4}\chi_0(\chi_0 + 2)$, which coincides with the Harish-Chandra homomorphism for $\mathfrak{sl}_2$, i.e. the formula expressing the value q_0 of the Casimir operator of $U(\mathfrak{sl}_2)$ on the Verma module with highest weight χ_0. This is a general property of the Miura transformation.

Simple analysis of the system (3.7) shows that it has exactly two solutions if $q_0 \neq m(m+2)/2$ for any $m \in \mathbb{Z}$. Each of the corresponding two solutions $\chi(t)$ of (3.5) gives rise to a solution $\phi(t) = e^{-\frac{1}{2}\int^t \chi(s)ds}$ of the equation

$$(3.8) \qquad (\partial_t^2 - q(t))\phi = 0.$$

Thus, if $q_0 \neq m(m+2)/4$ for any $m \in \mathbb{Z}$, then there are two linearly independent solutions of (3.8), and the monodromy matrix is semi-simple. In this case there are two equivalent ways to attach to each solution $\chi(t)$ an irreducible representation of $\widehat{\mathfrak{sl}}_2$ of critical level. One is to take the quotient of the Verma module $M_{\chi_0,-2}$ by the central character defined by $\partial_t^2 - q(t)$, and the other is to take the Wakimoto module $W_{\chi(t)}$.

Now suppose that $q_0 = m(m+2)/4$ for some $m \in \mathbb{Z}_+$. In this case there exists a unique solution of (3.5) the form $\chi(t) = -(m+2)/t + \dots$, but in general this is the only solution representable as a formal Laurent power series, and other solutions contain logarithmic terms. This means that the monodromy matrix for such $q(t)$ is the Jordan block with the eigenvalue ± 1.

If we set $\chi_0 = m \in \mathbb{Z}_+$, we can uniquely determine $\chi_n, -m \leq n \leq -1$, from equation (3.7). But then we obtain the equation $\sum_{i,j<0;i+j=-m-1} \chi_i \chi_j = 2q_{-m-1}$. It imposes a non-trivial condition on the coefficients of $q(t)$, which can be written as $P_m(q_{-1}, \dots, q_{-m-1}) = 0$, where P_m is a polynomial of degree $m+1$ if we put $\deg q_i = -i$. For example, $P_0 = q_{-1}$, $P_1 = 2q_{-1}^2 - q_{-2}$, etc. A solution of (3.5) of the form $\chi(t) = m/t + \dots$ exists if and only if the equation $P_m = 0$ is satisfied. In this case the projective connection $\partial_t^2 - q(t)$ has monodromy ± 1 around 0.

Now consider the Verma module $M_{m,-2}, m \in \mathbb{Z}_+$. Let $V_{m,-2}$ be the quotient of $M_{m,-2}$ by the submodule generated by the vector $f(0)^{m+1} v_{m,-2}$; in particular, $V_{0,-2}$ is the vacuum module V_{-2}. For $q(t) = m(m+2)/4t^2 + \dots$, denote by $V_m^{q(t)}$ the quotient of $V_{m,-2}$ by the central character corresponding to $\partial_t^2 - q(t)$.

Proposition 1. *The module $V_m^{q(t)}$ is non-zero if and only if the projective connection $\partial_t^2 - q(t)$ has monodromy ± 1.*

Proof. Suppose that $\partial_t^2 - q(t)$ has monodromy ± 1. Then there exists $\chi(t) = m/t + \dots$, which solves the Riccati equation (3.5). Consider the Wakimoto module $W_{\chi(t)}$. One can show that there exists a homomorphism $M_{m,-2} \to W_{\chi(t)}$, which maps $v_{m,-2}$ to $\mathbf{v}$ and $f(0)^{m+1} v_{m,-2}$ to 0. Hence we obtain a non-trivial homomorphism $V_{m,-2} \to W_{\chi(t)}$. But the center $Z(\widehat{\mathfrak{sl}}_2)$ acts on $W_{\chi(t)}$ according to the central character $q(t)$. Therefore this homomorphism factors through the homomorphism $V_m^{q(t)} \to W_{\chi(t)}$, and hence $V_m^{q(t)} \neq 0$. The "only if" part can proved in a similar fashion. $\square$

If the projective connection $q(t) = m(m+2)/4t^2 + \dots$ has monodromy ± 1, then there is a one-parameter family of solutions of the Riccati equation (3.5) of the form $\chi(t) = m/t + \dots$, and one solution of the form $\chi(t) = -(m+2)/t + \dots$. Thus, Wakimoto modules with central character $q(t)$ are parametrized by points of $\mathbb{P}^1$.

Remark. Consider the singular vector of the Virasoro algebra of type $(1, m+1)$. It is an element of the universal enveloping algebra of the subalgebra of the Virasoro algebra generated by $L_{-1}, L_{-2}, \dots$, depending on the central charge c. If we write it in terms of $L_i/c, i = -1, \dots, -m-1$, then all coefficients are polynomial in c^{-1}, cf. [33]. The classical limit of this vector is by definition the polynomial in $q_i \equiv L_i/c$, which is obtained from this expression by setting $c^{-1} = 0$. Using quantum Drinfeld-Sokolov reduction it is easy to show that this polynomial is equal to P_m (compare with [33]). $\square$

These results can be generalized for an arbitrary affine algebra $\widehat{\mathfrak{g}}$.

4. Geometric Langlands correspondence.

4.1. **Localization.** Let $\rho_p \in \mathcal{C}_+(\mathfrak{g}^L)$ be a $\mathfrak{g}^L$–oper on the formal disc around a point $p \in X$, and V^{ρ_p} be the $\widehat{\mathfrak{g}}$–module of critical level introduced in § 3.6. We want to

apply the localization functor from § 2 to the module V^{ρ_p}. It should give us a $\mathcal{D}_{-h^\vee}$–module $\Delta(V^{\rho_p})$ on the moduli space $\mathcal{M}_G(X)$. In fact, the square of the line bundle $\eta^{-h^\vee}$ is isomorphic to the canonical bundle ω on $\mathcal{M}_G(X)$. Hence $\mathcal{D}_{-h^\vee}$ is the sheaf of differential operators acting on $\omega^{1/2}$ [2]. Because of that we denote $\mathcal{D}_{-h^\vee}$ by $\mathcal{D}^{(1/2)}$.

Remark. The critical level is $-h^\vee$, because we use the invariant bilinear form on $\mathfrak{g}$, with respect to which the square of the maximal root of $\mathfrak{g}$ is equal to 2. If we were using the Killing form, the critical level would be $-1/2$. $\qquad\square$

Note that any regular $\mathfrak{g}^L$–oper on X can be restricted to the formal disc $\mathrm{Spec}\,\mathbb{C}[[t_p]]$.

Theorem 3 (A. Beilinson, V. Drinfeld [2])**.** *The $\mathcal{D}^{(1/2)}$–module $\Delta(V^{\rho_p})$ is non-zero if and only if ρ_p is the restriction of a regular $\mathfrak{g}^L$–oper ρ on X. In that case $\Delta(V^{\rho_p})$ is determined by ρ and does not depend on the position of p.*

According to [2], the $\mathcal{D}^{(1/2)}$–module $\Delta(V^{\rho_p})$ describes a system of differential equations on $\mathcal{M}_G(X)$, which is a quantization of Hitchin's system [37], cf. also [38]. Let $\mathcal{H}_G(X)$ be the moduli stack of Higgs pairs $\mathcal{H}_G(X)$, which is isomorphic to the cotangent stack of the moduli of G–bundles. Hitchin has defined a map $\nu : \mathcal{H}_G(X) \to \oplus_{i=1}^l H^0(X, \Omega^{d_i+1})$ and showed that it defines a completely integrable system on $\mathcal{H}_G(X)$.

By a quantization of the Hitchin system one should understand a commuting set of differential operators acting on a line bundle over $\mathcal{M}_G(X)$, whose symbols coincide with Hitchin's hamiltonians. Such differential operators can be constructed using the center $Z(\widehat{\mathfrak{g}})$ in the following way [2]. The algebra $U_{-h^\vee}(\widehat{\mathfrak{g}})$ and its local completion map to the ring of global differential operators acting on the line bundle $\xi^{-h^\vee}$ over $\mathcal{M}_G$, cf. § 2.1. The central elements from $Z(\widehat{\mathfrak{g}})$ define differential operators on $\mathcal{M}_G$, which commute with the right action of G_{in}. Hence one obtains a homomorphism $\jmath : Z(\widehat{\mathfrak{g}}) \to D^{(1/2)}$, where $D^{(1/2)}$ is the algebra of global differential operators acting on the line bundle $\omega^{1/2}$ over $\mathcal{M}_G(X) = \mathcal{M}_G/G_{\mathrm{in}}$.

Using the description of the center $Z(\widehat{\mathfrak{g}})$ provided by Theorem 2 and the description of the ring of regular functions on $\mathcal{H}_G(X)$ provided by Hitchin's theorem, Beilinson and Drinfeld [2] prove that if G is simply-connected, the homomorphism $\jmath$ is surjective. In particular, $D^{(1/2)}$ is commutative. Furthermore, $\mathrm{Spec}\,D^{(1/2)}$ is canonically isomorphic to the space of regular $\mathfrak{g}^L$–opers on X, and the symbol map gives an isomorphism $\mathrm{gr}\,D^{(1/2)} \simeq \mathbb{C}[\mathcal{H}_G(X)]$. Therefore elements of $D^{(1/2)}$ are indeed quantizations of the Hitchin hamiltonians.

Now each $\mathfrak{g}^L$–oper ρ on X provides a character of $D^{(1/2)}$, and hence a $\mathcal{D}^{(1/2)}$–module $\mathcal{D}^{(1/2)} \otimes_{D^{(1/2)}} \rho$. This $\mathcal{D}$–module is isomorphic to $\Delta(V^{\rho_p})$ defined above. It describes the system of differential equations

$$(4.1) \qquad\qquad \xi \cdot \Psi = \rho(\xi)\Psi, \qquad \xi \in D^{(1/2)}.$$

Thus, the construction outlined above associates to an arbitrary $\mathfrak{g}^L$–oper on X, a system of differential equations (a twisted $\mathcal{D}$–module) on $\mathcal{M}_G(X)$. Beilinson and Drinfeld put this construction in the context of Langlands correspondence.

4.2. The local Langlands correspondence.
The Langlands correspondence is a correspondence (bijection) between two different sets of objects associated to a field F and a reductive connected algebraic group G. Originally, it was formulated when F is

either a number field, or $F = \mathbb{F}_q(X)$, the field of rational functions on a smooth projective curve X defined over a finite field $\mathbb{F}_q$, as a far-reaching generalization of abelian class field theory (which corresponds to the simplest case when G is the multiplicative group) [1]. We will now briefly discuss some (elementary) aspects of the Langlands correspondence. This and the following subsections are written for non-experts and aim at giving them a very rough idea of a certain general picture which we will subsequently want to compare to the picture over $\mathbb{C}$. For more information we refer the reader to excellent reviews [39].

For simplicity we will restrict ourselves with the case of the function field $F = \mathbb{F}_q(X)$ and a (split connected) simple algebraic group G defined over $\mathbb{F}_q$. For a closed point $x \in X$ let $\mathcal{O}_x$ be the completion of the local ring of x, i.e. $\mathcal{O}_x \simeq \mathbb{F}_{q_x}[[t]]$, and $\mathcal{K}_x$ be its field of fractions, i.e. $\mathcal{K}_x \simeq \mathbb{F}_{q_x}((t))$, where $q_x = q^{\deg x}$.

Let $\bar{\mathbb{Q}}_l$ be the algebraic closure of the field $\mathbb{Q}_l$ of l–adic numbers, where l does not divide q. A representation of the group $G_x = G(\mathcal{K}_x)$ in a $\bar{\mathbb{Q}}_l$–vector space π_x is called smooth if the stabilizer of any vector of π_x is an open subgroup of G_x. A smooth irreducible representation is called *unramified* if there exists a non-zero vector $v_x \in \pi_x$ that is invariant with respect to the subgroup $K_x = G(\mathcal{O}_x)$; such a vector is then unique up to multiplication by $\bar{\mathbb{Q}}_l^\times$.

A complete description of the set of equivalence classes of irreducible unramified representations of G_x is known. They are parametrized by semi-simple conjugacy classes in the Langlands dual group G^L over $\bar{\mathbb{Q}}_l$. For the definition of the Langlands dual group, cf. [1, 39].[5] Here we will only say that the root and weight lattices of G and G^L are interchanged with the coroot and coweight lattices, respectively.

The correspondence between representations of G_x and conjugacy classes in $G^L(\bar{\mathbb{Q}}_l)$ can be described via the so-called *principal series* representations of G_x. It is easiest to define such a representation for $G = GL_n$. Any semi-simple conjugacy class in $G^L(\bar{\mathbb{Q}}_l) = GL_n(\bar{\mathbb{Q}}_l)$ contains a diagonal matrix $y = \operatorname{diag}(y_1, \ldots, y_n), y_i \in \bar{\mathbb{Q}}_l$, defined up to permutation. To such a class we can associate a character of the upper Borel subgroup

$$\chi_y(b) = (q_x^{1-n} y_1)^{\nu_x(b_{11})} (q_x^{2-n} y_2)^{\nu_x(b_{22})} \cdots y_n^{\nu_x(b_{nn})},$$

where b_{ii}'s are the diagonal entries of b, and ν_x is the standard norm on $\mathcal{K}_x$. This character then defines an induced representation of $GL_{n,x}$ in the space of locally constant functions $f(g)$ on $GL_{n,x}$, such that $f(bx) = \chi_y(b) f(g)$ for all $b \in B, f \in GL_{n,x}$. This is a principal series representation. It is known that it contains exactly one irreducible unramified component, which depends only on the conjugacy class of y. Thus one obtains a correspondence between conjugacy classes in $G^L(\bar{\mathbb{Q}}_l)$ and G_x–modules, which is called the local Langlands correspondence.

4.3. **Global Langlands correspondence.** Now suppose that we are given a set $\{y_x\}_{x \in X}$ of conjugacy classes of $G^L(\bar{\mathbb{Q}}_l)$ for all points of X. Then by the local Langlands correspondence we can associate to each of them a G_x–module, π_x.

Recall that the ring of adels $\mathbb{A}$ of F is the restricted product of all completions of F: $\mathbb{A} = \prod'_{x \in X} \mathcal{K}_x$. Restricted means that we only consider collections $(a_x)_{x \in X}$ for which $a_x \in \mathcal{O}_x$ for all but finitely many x. Note that F "diagonally" embeds into $\mathbb{A}$. The

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[5]a categorical definition, in the spirit of geometric Langlands correspondence, is given in [40]

group $G(\mathbb{A})$ is the restricted product of the groups $G_x, x \in X$, and we can consider its representation in the restricted tensor product $\otimes'_{x \in X} \pi_x$ of the modules π_x. The latter space is spanned by elements of the form $\otimes_{x \in X} w_x$, such that w_x is the K_x–invariant vector v_x for all but finitely many x. It is clear that the action of $G(\mathbb{A})$ on $\otimes'_{x \in X} \pi_x$ is well-defined.

The group $G(\mathbb{A})$ naturally acts on the space $C(G(F)\backslash G(\mathbb{A}))$ of locally constant functions on the quotient $G(F)\backslash G(\mathbb{A})$. An irreducible representation of $G(\mathbb{A})$ is called *automorphic* if it appears in the decomposition of $C(G(F)\backslash G(\mathbb{A}))$. Let us write such a representation as the restricted tensor product $\otimes_{x \in X} \pi_x$. Then it is called *unramified* if each π_x is unramified. In that case the tensor product of K_x–invariant vectors $\otimes_{x \in X} v_x$ is right-invariant with respect to the compact subgroup $K = \prod_{x \in X} K_x$. Hence it defines a function on the double quotient $G(F)\backslash G(\mathbb{A})/K$, which is called the *automorphic function* corresponding to $\otimes_{x \in X} \pi_x$. Note that $G(F)\backslash G(\mathbb{A})/K$ is the set of isomorphism classes of G–bundles on X.

A natural question is to describe all unramified automorphic representations of $G(\mathbb{A})$. In view of the local Langlands correspondence, we can pose this question in the following way: *for what sets $\{y_x\}_{x \in X}$ of conjugacy classes of $G^L(\bar{\mathbb{Q}}_l)$ is the corresponding representation $\otimes_{x \in X} \pi_x$ automorphic?*

Global Langlands correspondence gives (at least conjecturally) an answer to this question. It describes the "consistency conditions" on the set $\{y_x\}$, which make the representation $\otimes_{x \in X} \pi_x$ automorphic.

One approach to characterize them is analytic, using *L–functions*. To each local factor π_x and a finite-dimensional representation of G^L one can associate a local L–function. The product of these functions over all points of X is a global L–function of $\otimes_{x \in X} \pi_x$. It is known that if $\otimes_{x \in X} \pi_x$ is automorphic, then global L–function has analytic continuation and satisfies a functional equation (much as Dirichlet's L–functions) [1, 41, 42]. It turns out that for GL_2, the converse is also true: if the global L-function associated to the two-dimensional representation (and its twists by all continuous characters of $F^\times \backslash \mathbb{A}^\times$) has these nice properties, then the representation $\otimes'_{x \in X} \pi_x$ is automorphic [41]. A similar characterization of automorphic representations has also been obtained for other groups, cf. [39].

Another approach, which is closer to us, is to relate automorphic representations of $G(\mathbb{A})$ to representations of the *Galois group* $\mathrm{Gal}(\overline{F}/F)$. If we restrict ourselves with unramified representations of $G(\mathbb{A})$, we should consider the maximal unramified quotient of the Galois group, which is isomorphic to the fundamental group $\pi_1(X)$ of X. Actually, to be precise, we should consider the Weil group of F, which is the inverse image of $\mathbb{Z} \subset \hat{\mathbb{Z}} \simeq \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ under the homomorphism $\pi_1(X) \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, but we will ignore this subtlety.

To each point $x \in X$ one can associate canonically the *Frobenius* conjugacy class Fr_x of $\pi_1(X)$, cf. e.g. [43], p. 292. So any homomorphism $\sigma : \pi_1(X) \to G^L(\bar{\mathbb{Q}}_l)$ defines a collection $\{\sigma(\mathrm{Fr}_x)\}_{x \in X}$ of conjugacy classes in G^L. Now we can formulate the global Langlands conjecture.

Conjecture 1. *An irreducible unramified representation $\otimes'_{x \in X} \pi_x$ is automorphic if and only if there exists a continuous homomorphism $\sigma : \pi_1(X) \to G^L(\bar{\mathbb{Q}}_l)$, such that*

each π_x corresponds to the conjugacy class $\sigma(\mathrm{Fr}_x)$ in the sense of local Langlands correspondence.

In particular, this conjecture implies that there is a correspondence between the set of isomorphism classes of the irreducible unramified automorphic representations of $G(\mathbb{A})$ and the set of isomorphism classes of the homomorphisms $\pi_1(X) \to G^L(\bar{\mathbb{Q}}_l)$. However, in general this correspondence is not one-to-one, because there may be more than one homomorphism $\pi_1(X) \to G^L(\bar{\mathbb{Q}}_l)$ with the given set of the conjugacy classes $\sigma(\mathrm{Fr}_x)$ (on the automorphic side this should be reflected by the multiplicity of the corresponding representation in $C(G(F)\backslash G(\mathbb{A}))$ being greater than one). It is known that this can not happen for $G = GL_n$, and so in this case we should have a bijection. But for general G the picture is considerably more complicated.

4.4. **Local Langlands correspondence over** $\mathbb{C}$. The complex analogues of the groups G_x and K_x for the field $F = \mathbb{C}(X)$ are the loop group LG and its subgroup G_{in}. Define the category $\mathcal{O}^0_{\mathrm{crit}}$ of unramified $\widehat{\mathfrak{g}}$–modules of critical level, which consists of the modules, on which the action of $\mathfrak{g}_{\mathrm{in}}$ is locally finite, and which contain $\mathfrak{g}_{\mathrm{in}}$–invariant vector. On such modules, the action of the Lie algebra $\mathfrak{g}_{\mathrm{in}}$ can be integrated to an action of the Lie group G_{in}.

The analogue of a conjugacy class in the group G^L is a regular $\mathfrak{g}^L$–oper on the formal disc. (Note that the monodromy of a G^L–local system on a punctured disc gives rise to a conjugacy class in G^L; but of course all regular opers correspond to the class of the unit.)

The analogue of local Langlands correspondence will be the following: *each regular $\mathfrak{g}^L$–oper ρ_x on the formal disc defines an irreducible $\widehat{\mathfrak{g}}$–module of critical level.* This is the $\widehat{\mathfrak{g}}$–module V^{ρ_x} from § 3.6. Indeed, this module is irreducible, the action of the Lie algebra $\mathfrak{g}_{\mathrm{in}} = \mathfrak{g} \otimes \mathbb{C}[[t]]$ on it is locally finite, and it contains a unique G_{in}–invariant vector – the projection of the generating vector of V. Moreover, one can show that the modules V^{ρ_x} exhaust the irreducible objects of the category $\mathcal{O}^0_{\mathrm{crit}}$.

It is quite obvious that the Wakimoto modules are the analogues of representations of the principal series. For example, as we have seen in § 3.9, for each regular $\mathfrak{sl}_2$–oper on the disc, i.e. a projective connection $\partial_t^2 - q(t)$, where $q(t) = \sum_{n \leq -2} q_n t^{-n-2}$, there is a family of solutions $\chi(t)$ of the corresponding Riccati equation (3.5). These solutions are analogues of the character χ in the definition of principal series representation. The Wakimoto modules $W_{\chi(t)}$ corresponding to them contain a unique unramified component isomorphic to $V^{q(t)}$. So it depends on $q(t)$, but not on $\chi(t)$, just like an unramified representation depends on the conjugacy class of χ, but not on χ.

Remark. One can generalize the local Langlands correspondence for the group GL_n over a local field to include those representations, which are not unramified, but contain a fixed vector with respect to the Iwahori subgroup. Such representations correspond to a pair of semi-simple and unipotent conjugacy classes y and n in $G^L(\bar{\mathbb{Q}}_l)$, such that $yny^{-1} = n^{q_x}$ [44] (for a general group G one needs to introduce additional parameters).

The analogue of the Iwahori subgroup over $\mathbb{C}$ is the standard Borel subalgebra $\widetilde{\mathfrak{b}}_+$ of $\widehat{\mathfrak{g}}$, cf. § 3.1. Given a $\mathfrak{g}^L$–oper ρ on the disc with regular singularity in the sense of [6], we can construct a $\widehat{\mathfrak{g}}$–module of critical level M^ρ_χ, which has a vector invariant with

respect to $\widetilde{\mathfrak{b}}_+$. Here χ should be such that the most singular part of ρ is given by the central character on the Verma module M_χ over $\mathfrak{g}$, cf. § 3.6. But the module M_χ^ρ is not necessarily irreducible, and to specify its irreducible component, one has to fix an extra datum. For instance, in the case of $\mathfrak{g} = \mathfrak{sl}_2$, one has to fix in addition to a projective connection $\partial_t^2 - q(t)$, a solution of the equation (3.8) (or (3.5)). $\qquad\square$

4.5. **Global Langlands correspondence for curves over** $\mathbb{C}$**.** Now suppose that we are given a $\mathfrak{g}^L$–oper ρ_x for each point $x \in X$. We can assign to ρ_x the $\widehat{\mathfrak{g}}$–module of critical level V^{ρ_x}. Let $\mathfrak{g}(\mathbb{A}) = \prod'_{x \in X} \mathfrak{g}((t_x))$ be the restricted product of the loop algebras corresponding to the points of the curve, and $\widehat{\mathfrak{g}}(\mathbb{A})$ be its one-dimensional central extension, whose restriction to each factor coincides with the standard extension. The restricted tensor product $\otimes'_{x \in X} V^{\rho_x}$ is naturally a $\widehat{\mathfrak{g}}(\mathbb{A})$–module. Now we want to define an analogue of the automorphicity property for this module.

Recall that over a finite field an automorphic $G(\mathbb{A})$–module is the module that can be realized as a subspace of the space of functions on $G(F)\backslash G(\mathbb{A})$. Hence such a module defines an automorphic function on $G(F)\backslash G(\mathbb{A})/K$. Over the complex field we can not expect $\otimes'_{x \in X} V^{\rho_x}$ to be realized in the space of functions on $G(F)\backslash G(\mathbb{A})$. But we may hope to associate to it a multivalued function on $G(F)\backslash G(\mathbb{A})/K = \mathcal{M}_G(X)$, or, even better, a holonomic system of differential equations with regular singularities (which this function satisfies). The corresponding (twisted) $\mathcal{D}$–module on $\mathcal{M}_G(X)$ can be constructed by localization of the $\widehat{\mathfrak{g}}(\mathbb{A})$–module $\otimes'_{x \in X} V^{\rho_x}$.

The localization functor can be constructed in the same way as in § 2. The difference is that in § 2 we realized $\mathcal{M}_G(X)$ as $G_{\text{out}}\backslash LG/G_{\text{in}}$, using one point $p \in X$. Hence we assigned a twisted $\mathcal{D}$–module on $\mathcal{M}_G(X)$ to a single $\widehat{\mathfrak{g}}$–module attached to this point. Now we realize $\mathcal{M}_G(X)$ as $G(F)\backslash G(\mathbb{A})/K$ using all points of X. Hence to construct a $\mathcal{D}$–module, we have to attach a $\widehat{\mathfrak{g}}$–module M_x of some level k from the category $\mathcal{O}^0$ to each point of X.

Then we can assign a twisted $\mathcal{D}$–module on $G(F)\backslash G(\mathbb{A})$ to the $\widehat{\mathfrak{g}}(\mathbb{A})$–module $\otimes_{x \in X} M_x$ in the same way as in § 2. The action of $\mathfrak{g}_{\text{in}}$ on each M_x can be integrated to an action of the group G_{in}. Therefore our $\mathcal{D}$–module on $G(F)\backslash G(\mathbb{A})$ is K–equivariant and descends to a twisted $\mathcal{D}$–module on $\mathcal{M}_G(X)$, which we denote by $\widetilde{\Delta}(\otimes_{x \in X} M_x)$. One can show that $\widetilde{\Delta}(\otimes_{x \neq p} V_k \otimes M)$ (i.e. we attach a $\widehat{\mathfrak{g}}$–module M of level k to the point $p \in X$ and the vacuum module V_k to all other points) is isomorphic to $\Delta(M)$ defined in § 2.

Let us specialize to $k = -h^\vee$. For a given set $\{\rho_x\}_{x \in X}$ of local $\mathfrak{g}^L$–opers, let us call the $\widehat{\mathfrak{g}}(\mathbb{A})$–module $\otimes_{x \in X} V^{\rho_x}$ *weakly automorphic* if $\widetilde{\Delta}(\otimes_{x \in X} V^{\rho_x}) \neq 0$.

We can now state a weak version of global Langlands correspondence over $\mathbb{C}$: *the $\widehat{\mathfrak{g}}(\mathbb{A})$–module $\otimes_{x \in X} V^{\rho_x}$ is weakly automorphic if and only if there exists a globally defined regular $\mathfrak{g}^L$–oper ρ on X, such that for each $x \in X$, ρ_x is the restriction of ρ to a small disc around x.* This statement is analogous to Theorem 3. Moreover, if such ρ exists, then $\widetilde{\Delta}(\otimes_{x \in X} V^{\rho_x})$ is isomorphic to $\Delta(V^{\rho_p})$ from Theorem 3 for any $p \in X$. Let us denote this $\mathcal{D}^{(1/2)}$–module on $\mathcal{M}_G(X)$ attached to ρ by Δ_ρ.

On the other hand, one can attach to a regular $\mathfrak{g}^L$–oper on X a G^L–bundle with a connection, which is automatically integrable since $\dim X = 1$, and hence a homomorphism $\pi_1(X) \to G^L$. Therefore we obtain a correspondence between the homomorphisms $\pi_1(X) \to G^L$ corresponding to monodromies of $\mathfrak{g}^L$–opers on X and $\mathcal{D}$–modules on $\mathcal{M}_G(X)$, cf. [2].

4.6. **Hecke operators.** In the case of a finite field, the automorphic function on the double coset $G(F)\backslash G(\mathbb{A})/K$ corresponding to an automorphic representation $\otimes_{x \in X} \pi_x$ is an eigenfunction of the Hecke operators. Moreover, its eigenvalues are given by traces of $\sigma(\mathrm{Fr}_x)$ on the finite-dimensional representations of $G^L(\overline{\mathbb{Q}}_l)$, cf. [39].

It is possible to define analogues of Hecke operators over $\mathbb{C}$ as certain operations (correspondences) on $\mathcal{D}$–modules on $\mathcal{M}_G(X)$, cf. [11, 2]. Using these operations one can strengthen the statement of global Langlands correspondence by saying that Δ_ρ is an "eigensheaf" with respect to these Hecke correspondences, with "eigenvalues" given in terms of the G^L–local system defined by ρ [2].

More generally, one expects that for *any* homomorphism $\sigma : \pi_1(X) \to G^L(\mathbb{C})$ (equivalently, a G^L–local system on X, or a flat G^L–bundle over X), there exists a unique (up to isomorphism) $\mathcal{D}$–module $\mathcal{F}_\sigma$ on $\mathcal{M}_G(X)$, which is automorphic with respect to σ in the sense that it is an "eigensheaf" of Hecke correspondences with "eigenvalues" given in terms of σ. This is a general form of geometric Langlands correspondence over $\mathbb{C}$, which is due to Drinfeld.

If σ can be realized by a regular $\mathfrak{g}^L$–oper ρ on X, then Δ_ρ provides a candidate for $\mathcal{F}_\sigma$. Although not every G^L–local system can be realized by a regular $\mathfrak{g}^L$– oper, it can be realized by a $\mathfrak{g}^L$–oper with regular singularities at a finite number of points. Given such an oper, one can construct a candidate for $\mathcal{F}_\sigma$ by some modification of the localization functor defined above [2]. Unfortunately, σ can be realized by different $\mathfrak{g}^L$–opers, and so one has to check that the resulting $\mathcal{D}$–module depends only on σ.

5. Gaudin model and Bethe ansatz.

5.1. **Geometric Langlands correspondence in genus** 0. In this section we will consider $\mathcal{D}$–modules Δ_ρ when X has genus zero. In this case we will have to generalize the correspondence by allowing ramifications at a finite number of marked points. So, on the "Galois side" we consider G^L–opers on $\mathbb{P}^1$ with regular singularities at the marked points. On the "automorphic side", we consider twisted $\mathcal{D}$–modules on the moduli space of G–bundles with parabolic structures at the marked points. Recall that a parabolic structure on a G–bundle $\mathcal{P}$ at $p \in X$ is a choice of Borel subgroup in the fiber of $\mathcal{P}$ over p.

For convenience, let us choose a global coordinate t on $\mathbb{P}^1$, and let the marked points be $z_1, \ldots, z_N$, and ∞. We first consider the case of $\mathfrak{g} = \mathfrak{sl}_2$. On the "Galois side", we have projective connections on $\mathbb{P}^1$ with regular singularities at $z_1, \ldots, z_N$, and ∞. Such a connection has the form $\partial_t^2 - q(t)$ with

$$(5.1) \qquad q(t) = \sum_{i=1}^N \frac{c_i}{(t - z_i)^2} + \sum_{i=1}^N \frac{\mu_i}{t - z_i},$$

where $\sum_{i=1}^{N} \mu_i = 0$. On the "automorphic side" we consider systems of differential equations on the moduli space of rank two bundles over $\mathbb{P}^1$ with trivial determinant and with parabolic structures, i.e. a choice of a line in the fiber over each z_i and ∞. The set of such bundles is the double coset $G_{\text{out}} \backslash \prod_i LG_i / \prod_i \widetilde{B}_i$, where LG_i is the copy of the loop group of $G = SL_2$ associated to the ith marked point, $\widetilde{B}_i$ is its Borel subgroup, and G_{out} is the Lie group G over the ring of rational functions on $\mathbb{P}^1$, which are regular outside $z_1, \ldots, z_N$ and ∞.

The open subset $\mathcal{M}^{(N)}$ corresponding to the trivial bundle $\mathcal{O} \oplus \mathcal{O}$ is isomorphic to $(\mathbb{P}^1)^{N+1}/SL_2$: we choose lines in the two-dimensional fibers over $z_1, \ldots, z_N$ and ∞ up to a diagonal action of SL_2.

5.2. **Construction of $\mathcal{D}$–module.** Now introduce $\lambda_i, i = 1, \ldots, N$ and ∞, such that $\lambda_i(\lambda_i + 2)/4 = c_i, i = 1, \ldots, N$, and $\lambda_\infty(\lambda_\infty + 2)/4 = \sum_{i=1}^{N}(c_i + z_i \mu_i)$. Let $q_i(t - z_i), i = 1, \ldots, N$, and $q_\infty(t^{-1})$ be the expansions of $q(t)$ given by (5.1) around $z_i, i = 1, \ldots, N$, and ∞, respectively. Let $M_{\lambda_i}^{q_i}$ be the corresponding $\widehat{\mathfrak{sl}}_2$–modules of critical level introduced in § 3.6.

For $\lambda_1, \ldots, \lambda_N, \lambda_\infty \in \mathbb{Z}$, we have a line bundle $\boxtimes_{i=1,\ldots,N;\infty} \mathcal{O}(\lambda_i)$ over $(\mathbb{P}^1)^{N+1}$. This line bundle is equivariant with respect to the diagonal action of G. Denote by $\mathcal{L}$ the corresponding line bundle on $\mathcal{M}^{(N)}$, and let $\mathcal{D}_{\lambda_i}$ be the sheaf of differential operators on $\mathcal{L}$. A generalization of the localization functor from § 2 applied to the tensor product $\otimes_{i=1,\ldots,N;\infty} M_{\lambda_i}^{q_i}$ gives us a $\mathcal{D}_{\lambda_i}$–module on $\mathcal{M}^{(N)}$, which we denote by $\Delta(\lambda_i, \mu_i)$.

Let $H_i, i = 1, \ldots, N$, be the following elements of the algebra $U(\mathfrak{sl}_2)^{\otimes N}$:

$$(5.2) \qquad H_i = \sum_{j \neq i} \frac{e^{(i)} f^{(j)} + f^{(i)} e^{(j)} + \frac{1}{2} h^{(i)} h^{(j)}}{z_i - z_j},$$

where $a^{(i)}$ stands for the element $1 \otimes \ldots \otimes a \otimes \ldots \otimes 1$ with a in the ith place. The algebra $U(\mathfrak{sl}_2)$ maps to the algebra of global sections of the sheaf of differential operators on $\mathcal{O}(\lambda)$ (this algebra is well-defined for any λ). The elements H_i commute with the diagonal action of SL_2, and hence the corresponding differential operators lie in the algebra of global sections of $\mathcal{D}_{\lambda_i}$. One can also check directly that the operators H_i commute with each other. The following Proposition follows from Proposition 1 of [8].

Proposition 2. *The $\mathcal{D}_{\lambda_i}$–module $\Delta(\lambda_i, \mu_i)$ is isomorphic to the quotient of $\mathcal{D}_{\lambda_i}$ by the left ideal generated by $H_i - \mu_i, i = 1, \ldots, N$.*

Remark. An analogue of Theorem 3 is that the $\mathcal{D}_{\lambda_i}$–module obtained by localization of $\otimes_i M_{\lambda_i}^{q_i}$ is equal to 0, unless there exists $q(t)$ of the form (5.1), such that $q_i(t - z_i), i = 1, \ldots, N$, and $q_\infty(t^{-1})$ are the expansions of $q(t)$ around $z_i, i = 1, \ldots, N$, and ∞, respectively.

Note also that we have: $\sum_{i=1}^{N} H_i = 0$ and $\sum_{i=1}^{N} \mu_i = 0$, and that mutual commutativity of the operators H_i follows from Proposition 2. $\qquad \square$

5.3. **Gaudin model.** The $\mathcal{D}_{\lambda_i}$–module $\Delta(\lambda_i, \mu_i)$ describes the system of differential equations on $\mathcal{M}^{(N)}$:

$$(5.3) \qquad\qquad\qquad H_i \Psi = \mu_i \Psi.$$

This system of equations appears in the *Gaudin model* of statistical mechanics.

Gaudin's model [7] is a completely integrable quantum spin chain. We associate to the ith site of a one-dimensional lattice, a finite-dimensional $\mathfrak{sl}_2$–module V_{λ_i} of highest weight $\lambda_i \in \mathbb{Z}_+$ and a complex parameter $z_i, i = 1, \dots, N$. The space of states is the tensor product $\otimes_{i=1}^N V_{\lambda_i}$. The hamiltonians are the mutually commuting operators $H_i, i = 1, \dots, N$, given by (5.2), which act on the space of states. Gaudin's model can be considered as a degeneration of the XXX or XXZ Heisenberg magnetic chains, cf. [12]. In the latter the symmetry algebra is the Yangian of $\mathfrak{sl}_2$ or the affine quantum algebra $U_q(\widehat{\mathfrak{sl}}_2)$, respectively, while in Gaudin's model, it is the affine algebra $\widehat{\mathfrak{sl}}_2$.

One of the main problems arising in the Gaudin model is to find the joint spectrum of the operators H_i on $\otimes_{i=1}^N V_{\lambda_i}$. If we realize the operators H_i as differential operators on $\mathcal{M}^{(N)}$, then the diagonalization problem can be expressed as a system of differential equations (5.3) on $\mathcal{M}^{(N)}$. It is easy to write it down explicitly on an open subset of $\mathcal{M}^{(N)}$.

Let U_i be the big cell on the ith copy of $\mathbb{P}^1$ and let x_i be a coordinate on U_i. The differential operators on U_i corresponding to the basis elements of the ith copy of $\mathfrak{sl}_2$ are

$$(5.4) \qquad e^{(i)} = -x_i^2 \frac{\partial}{\partial x_i} + \lambda_i x_i, \quad h^{(i)} = 2x_i \frac{\partial}{\partial x_i} - \lambda_i, \quad f^{(i)} = \frac{\partial}{\partial x_i}.$$

Note that $\mathcal{M}^{(N)} \simeq \left(\mathbb{P}^1\right)^N / B$, where B is the Borel subgroup of SL_2. The system (5.3) can be considered as a system of B–invariant equations on $\left(\mathbb{P}^1\right)^N$. The restriction of this system to the product of U_i's, which is an open subset of $\left(\mathbb{P}^1\right)^N$, reads:

$$\sum_{j \neq i} \frac{1}{z_i - z_j} \left[-(x_i - x_j)^2 \frac{\partial^2}{\partial x_i \partial x_j} + (x_i - x_j) \left(\lambda_i \frac{\partial}{\partial x_j} - \lambda_j \frac{\partial}{\partial x_i} \right) + \frac{\lambda_i \lambda_j}{2} \right] \Psi$$

$$= \mu_i \Psi.$$

The $\mathcal{D}$–module $\Delta(\lambda_i, \mu_i)$ is non-trivial for *arbitrary* "eigenvalues" μ_i's, i.e. locally one can find solutions of the system above for any set of μ_i's. But these solutions generically have non-trivial monodromies around the diagonals.

In contrast, in Gaudin's model one is only interested in eigenvectors, which lie in the space of states $\otimes_{i=1}^N V_{\lambda_i}$. Such an eigenvector provides a *polynomial* solution of the system (4.1), which has no monodromies around the diagonals. This solution gives rise to a surjective homomorphism $\Delta(\lambda_i, \mu_i) \to \mathcal{L}$.

5.4. **Bethe ansatz.** The diagonalization problem for the Gaudin model (as well as for the majority of statistical models) is usually solved by the *Bethe ansatz* method. This method consists of the following.

There is an obvious eigenvector in $\otimes_{i=1}^N V_{\lambda_i}$: the tensor product $|0\rangle$ of the highest weight vectors v_{λ_i} of the V_{λ_i}'s. One constructs other eigenvectors by acting on this vector by certain elementary operators, depending on auxiliary parameters. For $w \in$

$\mathbb{C}, w \neq z_i$, and $a \in \{e, h, f\}$ put

$$(5.5) \qquad a(w) = \sum_{i=1}^{N} \frac{a^{(i)}}{w - z_i}.$$

Now introduce Bethe vectors

$$(5.6) \qquad |w_1, \dots, w_m\rangle = f(w_1) \dots f(w_m)|0\rangle \in \otimes_{i=1}^{N} V_{\lambda_i}.$$

Let us compute the action of Gaudin's hamiltonians on such a vector. It is convenient to pass to a family of operators

$$(5.7) \qquad S(t) = \sum_{i=1}^{N} \frac{\lambda_i(\lambda_i + 2)/4}{(t - z_i)^2} + \sum_{i=1}^{N} \frac{H_i}{t - z_i}, \qquad t \in \mathbb{C} \backslash \{z_1, \dots, z_N\}.$$

It is clear that the diagonalization problems for H_i's and $S(t), t \in \mathbb{C}$, are equivalent.

Note that Ψ is an eigenvector of the operators H_i with the eigenevalues μ_i if and only if $S(t)\Psi = q(t)\Psi$, where $q(t)$ is given by (5.1). Explicit computation shows that this formula holds for $\Psi = |w_1, \dots, w_m\rangle$ if and only if

$$(5.8) \qquad \sum_{i=1}^{N} \frac{\lambda_i}{w_j - z_i} - \sum_{s \neq j} \frac{2}{w_j - w_s} = 0, \qquad j = 1, \dots, m.$$

These equations are called *Bethe ansatz equations*.

If these equations are satisfied, then the Bethe vector (5.6) is an eigenvector, and also a highest weight vector of weight λ_∞ in $\otimes_{i=1}^{N} V_{\lambda_i}$. In this case the eigenvalue $q(t)$ can be represented as

$$(5.9) \qquad q(t) = \frac{1}{4}\chi(t)^2 - \frac{1}{2}\partial_t\chi(t),$$

where

$$(5.10) \qquad \chi(t) = \sum_{i=1}^{N} \frac{\lambda_i}{t - z_i} - \sum_{j=1}^{m} \frac{2}{t - w_j}.$$

The standard hypothesis is *completeness* of Bethe ansatz, which means that the Bethe vectors exhaust all the eigenvectors, which are highest weight vectors.

Now let us look at this hypothesis from the point of view of geometric Langlands correspondence. Consider the $\mathcal{D}_{\lambda_i}$–module obtained by localization of the tensor product of $\widehat{\mathfrak{sl}}_2$–modules $\otimes_{i=1,\dots,N;\infty} V_{\lambda_i}^{q_i}$, where q_i's were defined in § 5.2, and V_λ^q was defined in § 3.9. The fibers of this $\mathcal{D}_{\lambda_i}$ over all points of $\mathcal{M}^{(N)}$ are isomorphic to the space of coinvariants $\mathcal{H} = (\otimes_{i=1,\dots,N;\infty} V_{\lambda_i}^{q_i})/\mathfrak{g}_{\text{out}}$. But $(\otimes_{i=1,\dots,N;\infty} V_{\lambda_i,-2})/\mathfrak{g}_{\text{out}} \simeq (\otimes_{i=1,\dots,N;\infty} V_{\lambda_i})^G$, cf. e.g. [8]. Therefore $\mathcal{H}$ is isomorphic to the quotient of $(\otimes_{i=1,\dots,N;\infty} V_{\lambda_i})^G$ by the action of the operators $H_i - \mu_i, i = 1, \dots, N$.

If there exists an eigenvector of the Gaudin operators H_i's with the eigenvalues μ_i's in $(\otimes_{i=1,\dots,N;\infty} V_{\lambda_i})^G$, then the space $\mathcal{H}$ is non-zero. But then $V_{\lambda_i}^{q_i} \neq 0$ for all $i = 1, \dots, N; \infty$. By Proposition 1, this implies the following result.

Proposition 3. *If there is an eigenvector of the Gaudin hamiltonians in $\otimes_{i=1}^{N} V_{\lambda_i}$ with the eigenvalues $\mu_i, i = 1, \ldots, N$, then all solutions of the differential equation*

$$(5.11) \qquad \left(\partial_t^2 - \sum_{i=1}^{N} \frac{\lambda_i(\lambda_i + 2)/4}{(t - z_i)^2} - \sum_{i=1}^{N} \frac{\mu_i}{t - z_i} \right) \phi = 0$$

on $\mathbb{P}^1$ have monodromies ± 1 around $z_1, \ldots, z_N$ and ∞, and hence the monodromy representation of the projective connection $\partial_t^2 - q(t)$ defining the $\mathcal{D}_{\lambda_i}$–module $\Delta(\lambda_i, \mu_i)$ is trivial.

But this property of (5.11) is equivalent to the Bethe ansatz equations! Indeed, if the equation (5.11) has monodromies ± 1, then there is a solution of the form

$$(5.12) \qquad \phi(t) = \prod_{i=1}^{N} (t - z_i)^{-\lambda_i/2} \prod_{j=1}^{m} (t - w_j),$$

for some $w_1, \ldots, w_m \in \mathbb{C}$. But then formulas (5.9) and (5.10) must hold. The left hand side of (5.9) has no poles at the points w_j's. Therefore there should be no poles at w_j's in the right hand side. Straightforward computation shows that this condition is equivalent to Bethe ansatz equations (5.8). In other words, Bethe ansatz equations mean that the Miura transformation (5.9) "erases" the extra poles of the connection $\partial_t - \chi(t)/2$ given by (5.10).

On the other hand, if Bethe ansatz equations (5.8) are satisfied for some $w_1, \ldots, w_m$, then formula (5.9) holds. Hence (5.12) is a solution of the equation (5.11). A linearly independent solution can be constructed as $\phi(t) \int^t \phi(w)^{-2} dw$, and it is clear that both of them have monodromies ± 1.

Thus, if there is an eigenvector of the Gaudin hamiltonians with the eigenvalues $\mu_i, i = 1, \ldots, N$, then there exist $w_1, \ldots, w_m$ such that equations (5.9) and (5.8) are satisfied. But then we can construct this eigenvector explicitly as the Bethe vector (5.6). The completeness of Bethe ansatz now boils down to the linear independence of these vectors, which can be shown by elementary methods for generic values of z_i's.

We conclude that Bethe ansatz describes those $\mathcal{D}_{\lambda_i}$–modules $\Delta(\lambda_i, \mu_i)$ on $\mathcal{M}^{(N)}$, which admit a surjective homomorphism to an invertible sheaf. It tells us that the $\mathfrak{sl}_2$–opers, to which they correspond, generate trivial monodromy representations $\pi_1(\mathbb{P}^1 \backslash \{z_1, \ldots, z_N, \infty\}) \to PGL_2$.

5.5. Generalization to other Lie algebras.
Starting from a $\mathfrak{g}^L$–oper on $\mathbb{P}^1$ with regular singular points $z_1, \ldots, z_N$ and ∞, one can define a twisted $\mathcal{D}$–module on the moduli space of G–bundles on $\mathbb{P}^1$ with parabolic structures at $z_1, \ldots, z_N$ and ∞.

The open subset of the moduli space corresponding to the trivial G–bundle is isomorphic to F^{N+1}/G. The restriction of our twisted $\mathcal{D}$–module to this subset is the quotient of the appropriate sheaf of differential operators on F^{N+1}/G by the left ideal generated by the images of the central elements from $Z(\widehat{\mathfrak{g}})$. It is easy to write down a formula for the generators of the ideal corresponding to the Sugawara elements: we just have to replace the numerator in (5.2) by $\sum_a J_a^{(i)} J_a^{(j)}$. These operators are the hamiltonians of the Gaudin model associated to $\mathfrak{g}$. The new fact is that in general the subalgebra of $U(\mathfrak{g})^{\otimes(N+1)}$ of elements commuting with H_i's and among themselves is

quite large, because it includes elements corresponding to higher order central elements from $Z(\widehat{\mathfrak{g}})$.

The existence of an eigenvector of the generalized Gaudin hamiltonians is equivalent to the existence of a surjective homomorphism from the corresponding $\mathcal{D}$–module to an invertible sheaf on F^{N+1}/G. A straightforward generalization of our argument from § 5.4 implies that such a homomorphism exists only if the monodromy representation of the corresponding $\mathfrak{g}^L$–oper on $\mathbb{P}^1$ is trivial. The latter condition can be expressed as a system of algebraic equations similar to the Bethe ansatz equations for $\mathfrak{g} = \mathfrak{sl}_2$. If these equations are satisfied, one can construct explicitly the corresponding eigenvectors, which look similar to the Bethe vectors for $\mathfrak{g} = \mathfrak{sl}_2$. The completeness of Bethe ansatz then follows from linear independence of these vectors.

In [8] eigenvectors of generalized Gaudin hamiltonians were constructed explicitly using Wakimoto modules of critical level. Another construction was given in [46]. The formula for these eigenvectors was suggested in [47], though the "off-shell Bethe ansatz equations" of [47] had already been proved in [48].

Let us explain in more detail the case of $\mathfrak{g} = \mathfrak{sl}_3$; the general case is similar.

5.6. **The case of $\mathfrak{sl}_3$.** We start with a $\mathfrak{sl}_3$–oper ρ, which is a third order differential operator on $\mathbb{P}^1$ with regular singularities at $z_1, \dots, z_N$, and ∞:

$$(5.13) \quad \rho = \partial_t^3 - \sum_{i=1}^{N} \left(\frac{c^1(\lambda_i)}{(t - z_i)^2} + \frac{\mu_i}{t - z_i} \right) \partial_t - \sum_{i=1}^{N} \left(\frac{c^2(\lambda_i)}{(t - z_i)^3} + \frac{\nu_i}{(t - z_i)^2} + \frac{\kappa_i}{t - z_i} \right).$$

To construct the corresponding $\mathcal{D}$–module, we have to choose the weights $\lambda_1, \dots, \lambda_N$ and λ_∞ of $\mathfrak{sl}_3$, such that $c^1(\lambda)$ and $c^2(\lambda)$ are the values of the central elements C^1 and C^2 of $U(\mathfrak{sl}_3)$ of orders 2 and 3, respectively, on the Verma module M_λ. We will assume that $\lambda_1, \dots, \lambda_N, \lambda_\infty$ are integral dominant weights. Let

$$\rho_i = \partial_t^3 - q_i^1(t - z_i)\partial_t + q_i^2(t - z_i), \qquad i = 1, \dots, N,$$

be the expansions of the oper (5.13) at z_i, and ρ_∞ be its expansion at ∞. Each ρ_i defines a character of the center $Z(\widehat{\mathfrak{sl}_3})$. Recall that in this case the center is generated by the Fourier components of the Sugawara field $S^1(z)$ of order 2 and a field $S^2(z)$ of order 3. The values of the central character on them are given by the Fourier components of q_i^1 and q_i^2, respectively. Note that $c^1(\lambda_i)$ and μ_i are the values on the components S_0^1 and S_{-1}^1, respectively, while $c^2(\lambda_i), \nu_i$ and κ_i are the values on the components S_0^2, S_{-1}^2 and S_{-2}^2, respectively.

To each marked point we can now associate a $\widehat{\mathfrak{g}}$–module of critical level, $M_{\lambda_i}^{\rho_i}$, and apply the localization functor to the tensor product of these modules. This gives us a $\mathcal{D}$–module on F^{N+1}/SL_3. The corresponding system of differential equations consists of N second order differential equations, and $2N$ third order differential equations.

The existence of an eigenvector of the generalized Gaudin hamiltonians in $\otimes_{i=1}^{N} V_{\lambda_i}$ implies, in the same way as in the case $\mathfrak{g} = \mathfrak{sl}_2$, that the monodromy representation $\pi_1(\mathbb{P}^1 \backslash \{z_1, \dots, z_N, \infty\}) \to PGL_3$ defined by the oper (5.13) is trivial.

But the monodromy is trivial if and only if this oper admits a global Miura transformation:

$$(5.14) \qquad \rho = (\partial_t - \chi_1(t))(\partial_t - \chi_2(t))(\partial_t - \chi_3(t)),$$

where $\chi_i(t)$ are the components of a diagonal connection $\partial_t + \chi(t)$.

To see that, observe that it follows from formulas (5.13) and (5.14) that $\chi(t)$ must be of the form

$$(5.15) \qquad \chi(t) = \sum_{i=1}^{N} \frac{\lambda_i}{t - z_i} + \sum_{s \in W; s \neq 1} \sum_{j_s=1}^{m_s} \frac{s(\bar{\rho}) - \bar{\rho}}{t - w_{j_s}^{(s)}},$$

where W is the Weyl group of $\mathfrak{sl}_3$. Here the weights of $\mathfrak{sl}_3$ are represented by diagonal matrices, so that the fundamental weights are represented as $\omega_1 = \mathrm{diag}[2/3, -1/3, -1/3]$ and $\omega_2 = \mathrm{diag}[1/3, 1/3, -2/3]$.

Indeed, it is clear from (5.14) that if $\chi(t)$ has a pole different from $z_1, \ldots, z_N$, then it can not be of order greater than 1. If such an extra singular term of the form $\gamma/(t-w)$ exists for $w \neq z$, then $c^1(\gamma) = c^2(\gamma) = 0$, because ρ is assumed to be regular at w. But the weights which satisfy this condition lie in the orbit of 0 with respect to the action of the Weyl group W of $\mathfrak{sl}_3$ shifted by the half-sum of the positive roots $\bar{\rho}$. For $\mathfrak{g} = \mathfrak{sl}_3$ these weights are: $0, -\alpha_1, -\alpha_2, -2\alpha_1 - \alpha_2, -\alpha_1 - 2\alpha_2, -2\alpha_1 - 2\alpha_2$.

If ρ can be represented in the form (5.14) with λ_i's integral dominant weights, then it is easy to find linearly independent solutions of the equation $\rho \cdot \phi = 0$:

$$e^{\int^t \chi_3(u)du}, \qquad e^{\int^t \chi_3 du} \int^t dx e^{\int^x (\chi_2 - \chi_3)du},$$

$$e^{\int^t \chi_3 du} \int^t dx \, e^{\int^x (\chi_2 - \chi_3)du} \int^x dy \, e^{\int^y (\chi_1 - \chi_2)du}.$$

It follows from these formulas that in the neighborhood of each point z_i local solutions look as follows: $(t - z_i)^{(-n_i^1 - 2n_i^2)/3}(1 + \ldots)$, $(t - z_i)^{(n_i^2 - n_i^1)/3+1}(1 + \ldots)$, and $(t - z_i)^{(2n_i^1 + n_i^2)/3+2}(1 + \ldots)$, where we put $\lambda_i = n_i^1 \omega_1 + n_i^2 \omega_2$. Hence the monodromy of each of these solutions around z_i is a third root of unity, i.e. the identity in PGL_3.

On the other hand, if ρ has a trivial monodromy representation, then it has three linearly independent global solutions, and hence it is representable in the form (5.14).

Remark. Recall that ρ admits a Miura transformation if and only if it admits a flag of invariant subbundles, i.e. the underlying rank 3 bundle E has invariant subbundles E_1' and E_2' of ranks 1 and 2. But the oper structure already defines subbundles, E_1 and E_2 of ranks 1 and 2, cf. § 3.5. Thus, the fiber of E at each point of $\mathbb{P}^1 \backslash \{z_1, \ldots, z_N, \infty\}$ contains two flags. The function $\chi(t)$ has a singularity at a point w if these two flags are not in generic position at w. Non-generic positions of two flags are labeled by elements of the Weyl group. Hence we obtain a summation over the Weyl group in (5.15). $\qquad \square$

5.7. **Generalized Bethe equations.** Representability of ρ in the form (5.14) is equivalent to a system of algebraic equations on the positions of the points $w_{j_s}^{(s)}$. Indeed, we have already made the most singular terms of the right hand side of (5.14) vanish, but vanishing of other singular terms imposes additional relations. Suppose for simplicity

that in (5.15) only the simple reflections s_i occur. Since $s_i(\bar\rho) - \bar\rho = -\alpha_i$, we have in this case:

$$(5.16) \qquad \chi(t) = \sum_{i=1}^{N} \frac{\lambda_i}{t - z_i} - \sum_{j=1}^{m} \frac{\alpha_{i_j}}{t - w_j}.$$

According to Proposition 6 in [8] the corresponding equations read:

$$(5.17) \qquad \sum_{i=1}^{N} \frac{(\lambda_i, \alpha_{i_j})}{w_j - z_i} - \sum_{l \neq j} \frac{(\alpha_{i_l}, \alpha_{i_j})}{w_j - w_l} = 0, \quad j = 1, \dots, m.$$

They can be called the generalized Bethe ansatz equations for $\mathfrak{g} = \mathfrak{sl}_3$. Analogous equations can be written for an arbitrary Lie algebra $\mathfrak{g}$, cf. [8].

The equations (5.17) have been interpreted in [8] as the equations on the existence of a null-vector (cf. § 3.1) in the Wakimoto module $W_{\chi^j(t)}$, where $\chi^j(t - w_j)$ is the expansion of $\chi(t)$ at w_j for $i = 1, \dots, m$. If equations (5.17) hold, then such null-vectors exist and using them one can construct explicitly an eigenvector of the Gaudin hamiltonians, cf. Theorem 3 in [8]. The formula for this eigenvector and the equations (5.17) can also be derived from the asymptotic analysis of solutions of the Knizhnik-Zamolodchikov equation [46], cf. also [47, 48].

5.8. **Remarks.** 1. Formula (5.12) shows that in the case $\mathfrak{g} = \mathfrak{sl}_2$ the numbers w_j's defining the Bethe vectors can be found as the zeroes of the polynomial $\phi \prod_{i=1}^{N}(t - z_i)^{\lambda_i/2}$, where ϕ is a solution of the differential equation (5.11) (the one for which this expression is really a polynomial).

In general one can find the numbers w_j's in a similar way. Let us explain this for $\mathfrak{g} = \mathfrak{sl}_3$ in the case when we only have degeneracies corresponding to the simple reflections. Then the positions of the degeneracies corresponding to the simple reflection s_2 are the zeroes of the polynomial $\phi \prod_{i=1}^{N}(t - z_i)^{(n_i^1 + 2n_i^2)/3}$, where ϕ is a solution of the equation $\rho \cdot \phi = 0$. On the other hand, the positions of the degeneracies corresponding to the simple reflection s_1 are the zeroes of the polynomial $\widetilde\phi \prod_{i=1}^{N}(t - z_i)^{(2n_i^1 + n_i^2)/3}$, where $\widetilde\phi$ is a solution of the equation $\rho^{\mathrm{ad}} \cdot \widetilde\phi = 0$. Here ρ^{ad} stands for the adjoint of the differential operator ρ given by (5.13). This result has been independently obtained by A. Varchenko by other methods. The appearance of Miura transformation in Gaudin models has also been studied in [49] from a different point of view.

2. If we allow singular points corresponding to other elements of the Weyl group, we obtain more complicated equations of the type (5.17) for these points. These equations can also be interpreted as the equations on the existence of a null-vector in the Wakimoto module corresponding to the expansion of $\chi(t)$ at such a point. If the equations hold, then the null-vector exists, and we can construct an eigenvector of the Gaudin hamiltonians in the same way as in [8], § 5.

Presumably, for generic $z_1, \dots, z_N$ all eigenvectors of the Gaudin hamiltonians correspond to the simplest degeneracies of the opers, i.e. only those labeled by the simple reflections from the Weyl group. But for special values of $z_1, \dots, z_N$, some of the eigenvectors may correspond to other degeneracies.

3. When X is of genus one, the $\mathcal{D}$–modules obtained by geometric Langlands correspondence can also be described explicitly. As special cases one obtains the Calogero-Sutherland systems with elliptic potential [45].

6. Drinfeld's construction and Sklyanin's separation of variables.

V. Drinfeld has given another construction of geometric Langlands correspondence in the case of GL_2 [10]. Recall from § 4.3 that the goal is to attach to a two-dimensional l–adic representation σ of $\pi_1(X)$, an automorphic function on $G(F)\backslash G(\mathbb{A})/K$. Such a function can be viewed as a $G(F)$–invariant function on $G(\mathbb{A})/K$. H. Jacquet and R.P. Langlands [41] constructed a function on $G(\mathbb{A})/K$, which is $B(F)$–invariant, where B is the Borel subgroup of G. It remains then to establish that it is actually $G(F)$–invariant. H. Jacquet and R.P. Langlands derived this from the functional equation on the corresponding L–function (which they had assumed from the beginning).

V. Drinfeld's approach [10] (cf. also [11]) was to associate a perverse sheaf to the function constructed by Jacquet-Langlands, and show that this sheaf is constant along the fibers of the projection $G(F)\backslash G(\mathbb{A})/K \to B(F)\backslash G(\mathbb{A})/K$.

6.1. **Whittaker function.** The Jacquet-Langlands function is constructed using the *Whittaker model.* Let G be a simple Lie group. Choose a non-trivial additive character $\Phi : \mathbb{A}/F \to \bar{\mathbb{Q}}_l^{\times}$. Let W be the representation of $G(\mathbb{A})$ in the space of locally constant functions $f : G(\mathbb{A}) \to \bar{\mathbb{Q}}_l$, such that

$$(6.1) \qquad f(u^{-1}x) = \sum_{i=1}^{l} \Phi(u_i) f(x)$$

for any element u of the unipotent subgroup $N(\mathbb{A}) \simeq \mathbb{A}$. Here u_i is the element of u corresponding to the ith simple root of G. A Whittaker model for a $G(\mathbb{A})$–module M is by definition a non-trivial homomorphism $M \to W$.

Recall that by local Langlands correspondence we can associate to a homomorphism $\sigma : \pi_1(X) \to G^L$ a collection of unramified representations π_x of the groups $G_x, x \in X$. It is known that the $G(\mathbb{A})$–module $\otimes_{x \in X} \pi_x$ has a unique Whittaker model if all factors π_x are non-degenerate [50]. The image of the vector $\otimes_{x \in X} v_x$ under the homomorphism to W is a function ϕ_σ on $G(\mathbb{A})$, which is called *Whittaker function.* This function is right invariant with respect to K and satisfies equation (6.1).

It is easy to see that $N(\mathcal{K}_x)\backslash G_x/G(\mathcal{O}_x)$ is isomorphic to the root lattice of G and hence to the weight lattice $P^{\vee}$ of the Langlands dual group G^L. Therefore $N(\mathbb{A})\backslash G(\mathbb{A})/K$ is isomorphic to the set $\mathrm{Div}_{P^\vee}$ of $P^{\vee}$–valued divisors on X. Hence the Whittaker function ϕ_σ can be considered as a function on $\mathrm{Div}_{P^\vee}$. It turns out [51] that its values on non-effective divisors are equal to 0, and

$$(6.2) \qquad \phi_\sigma \left(\sum_i \lambda_i [x_i] \right) = \prod_i \mathrm{tr}\ \sigma(\mathrm{Fr}_{x_i})|_{V_{\lambda_i}}, \qquad \lambda_i \in P_+^\vee,$$

up to a power of q. Here V_λ is the finite-dimensional representation of G^L of highest weight $\lambda \in P_+^\vee$.

6.2. **Geometric interpretation.** From now on we put $G = GL_2$. Then σ can be considered as an irreducible two-dimensional representation of $\pi_1(X)$. This representation corresponds to a rank two locally constant sheaf E_σ on X, cf. e.g. [43]. Since $P^\vee = \mathbb{Z}$ in this case, $\mathrm{Div}_{P^\vee}$ is just the set of divisors on X. The function ϕ_σ is zero away from the subset Div_+ of effective divisors, and its values on Div_+ are given by formula (6.2).

The set of effective divisors on X of degree m is the set of points of the mth symmetric power of X, $S^m X$. Consider the irreducible perverse sheaf $E_\sigma^{(m)} = \left(\pi_* E_\sigma^{\boxtimes m}\right)^{S_m}$ on $S^m X$, where $\pi : X^m \to S^m X$ is the projection. The stalk of $E_\sigma^{(m)}$ over a divisor $\sum_i n_i[x_i]$, where x_i's are distinct, is $\otimes_i S^{n_i} E_{\sigma,x}$. Let ϕ_σ^m be the restriction of ϕ_σ to $S^m X$. It follows from (6.2) that the value of ϕ_σ^m at $d \in S^m X$ is given by the trace of the Frobenius element of d on the stalk of $E_\sigma^{(m)}$ over d.

Thus, geometrically, the passage from σ to the Whittaker function is the passage from E_σ to $E_\sigma^{(m)}, m > 0$. This is the first step of Drinfeld's construction [10].

Now consider the function

$$(6.3) \qquad \widetilde{g}_\sigma(x) = \sum_{a \in F^\times} \phi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} x\right).$$

Formula (6.1) implies that this function is left invariant with respect to $B(F)$, and hence it gives rise to a function g_σ on $B(F)\backslash G(\mathbb{A})/K$. This is the Jacquet-Langlands function.

Drinfeld has interpreted the space $B(F)\backslash G(\mathbb{A})/K$ as the set of points of the moduli space $\mathcal{M}_{2,1}$ of rank two bundles on X with a rank one subbundle. An element of $\mathcal{M}_{2,1}$ can be considered after twisting by an appropriate line bundle as an extension

$$(6.4) \qquad 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{L}_2 \longrightarrow \mathcal{L}_1 \longrightarrow 0.$$

Let $\mathcal{M}_{2,1}^n$ be the component of $\mathcal{M}_{1,2}$, which consists of extensions (6.4) where $\mathcal{L}_1$ and hence $\mathcal{L}_2$ are of degree n. There is a map $j_n^\vee : \mathcal{M}_{2,1}^n \to Jac_n$ to the nth Jacobean variety Jac_n of X, which sends an extension (6.4) to the class of $\mathcal{L}_1$. If $n > 2g - 2$, $j_n^\vee$ is a projective fibration with the fiber $\mathbb{P}H^1(X, \mathrm{Hom}(\mathcal{L}_1, \mathcal{O}))$.

The dual fibration is isomorphic to $S^m X$, where $m = n + 2g - 2$. Indeed, there is natural map $j_n : S^m X \to Jac_n$, which sends a divisor $d \in S^m$ to $\mathcal{O}(d) \otimes \Omega^{-1} \in Jac_n$, where Ω is the canonical bundle. The fiber of the map j_n over $\mathcal{L}_1$ is $\mathbb{P}H^0(X, \mathcal{L}_1 \otimes \Omega)$, which is dual to the fiber of $j_n^\vee$ over $\mathcal{L}_1$, by the Serre duality.

For a pair of dual projective bundles over the same base one can define *Radon transform*, which maps functions on one of them to functions on the other. As shown in [10], formula (6.3) actually means that the restriction g_σ^n of g_σ to $\mathcal{M}_{2,1}^n$ is the Radon transform of the function ϕ_σ^m on $S^m X$.

One can define the Radon transform on perverse sheaves, cf. e.g. [52, 24, 11]. It follows from a theorem of P. Deligne, cf. [10], that the Radon transform of $E_\sigma^{(m)}$ is an irreducible perverse sheaf on $\mathcal{M}_{2,1}^n$, which we denote by $\mathcal{G}_\sigma^n$. The sheaf $\mathcal{G}_\sigma^n$ is the geometric object, which replaces the function g_σ^n in the sense that the value of g_σ^n at a point p of $\mathcal{M}_{2,1}^n$ is given by the trace of the Frobenius element of p on the fiber of $\mathcal{G}_\sigma^n$ over p.

$$\mathcal{M}_{2,1}^n \qquad\qquad\qquad S^m X$$

$$i_n \qquad\qquad j_n^\vee \qquad\qquad j_n$$

$$\mathcal{M}_2^n \qquad\qquad\qquad Jac_n$$

6.3. Descent.

There is a natural projection $i_n : \mathcal{M}_{2,1}^n \to \mathcal{M}_2^n$ to the moduli space of rank two bundles on X of degree n, which maps an extension (6.4) to the class of $\mathcal{L}_2$. This is of course just the geometric realization of the projection $G(F)\backslash G(\mathbb{A})/K \to B(F)\backslash G(\mathbb{A})/K$. The fiber of i_n over $\mathcal{L}_2 \in \mathcal{M}_2^n$ is the set of non-vanishing sections of $\mathcal{L}_2$ up to a non-zero multiple, which is the complement of a divisor in $\mathbb{P}H^0(X, \mathcal{L}_2)$.

Remark. Here we consider i_n as a map of stacks. But we could also view it as a rational map from the moduli space of semi-stable pairs (bundle,section) to the moduli space of semi-stable bundles. In this setting, beautiful resolutions of this map have been given by A. Bertram [53] and M. Thaddeus [54] (cf. also [55]). □

Drinfeld proves that $\mathcal{G}_\sigma^n$ is constant along the generic fiber of i_n. This implies that $\mathcal{G}_\sigma^n$ is a pull-back of a perverse sheaf $\mathcal{F}_\sigma^n$ on $\mathcal{M}_2^n$ for all $n > 2g - 2$. After that Drinfeld constructs the automorphic function as the function attached to the sheaves $\mathcal{F}_\sigma^n$; using the Hecke operators this function can be uniquely extended to the whole $\mathcal{M}_2$ [10].

6.4. Comparison of two constructions.

Since the construction outlined above is purely geometric, it makes sense over the field of complex numbers as well, and it establishes geometric Langlands correspondence for GL_2 in the sense of § 4.5 (more precisely, in one direction, while over $\mathbb{F}_q$ this has also been done in the opposite direction, cf. [56]). It is interesting to compare it with the construction outlined in § 4.5. Over the field of complex numbers, one can switch from perverse sheaves to $\mathcal{D}$–modules using the Riemann-Hilbert correspondence, cf. e.g. [24].

Let us also switch from GL_2 to SL_2: we have $\mathcal{M}_{SL_2}(X) \simeq \mathcal{M}_2^n$ for even n. Let us take as $\mathcal{F}_\sigma^n$ the $\mathcal{D}$–module Δ_ρ obtained from a particular projective connection ρ, whose monodromy representation coincides with $\sigma : \pi_1(X) \to PGL_2$. Now take its inverse image $i_n^* \delta(\rho)$ on $\mathcal{M}_{2,1}^n$ and apply the inverse Radon transform. This gives us a $\mathcal{D}$–module on $S^m X$, where $m = n + 2g - 2$. The anticipated equivalence of the two constructions simply means that the latter is just the $\mathcal{D}$–module corresponding to $E_\sigma^{(m)}$ by the Riemann-Hilbert correspondence (i.e. that $E_\sigma^{(m)}$ is the sheaf of solutions of this $\mathcal{D}$–module). Clearly, this would imply that Δ_ρ is automorphic and depends on σ, but not on ρ, and that $\mathcal{G}_\sigma$ is constant along the fibers.

In order to establish the equivalence of the two constructions, it is important to understand what is the analogue of the Whittaker model over the complex field. The existence of the Whittaker model over a finite field is equivalent to the existence of a non-trivial coinvariant of the unipotent subgroup N in $M \otimes \Phi^*$, where M is a G–module and Φ^* is the character of N dual to Φ.

Now let M be a $\widehat{\mathfrak{g}}$–module from the category $\mathcal{O}_{\mathrm{crit}}^0$, and χ be a character of $L\mathfrak{n}_+$. A naive analogue of the local Whittaker model is the space of coinvariants of $M \otimes \chi$ with respect to the Lie subalgebra $L\mathfrak{n}_+$. But this space is zero, because the subalgebra

$\mathfrak{n}_+ \otimes \mathbb{C}[t]$ of $L\mathfrak{n}_+$ acts locally nilpotently on M. The correct analogue is the *semi-infinite* cohomology of $L\mathfrak{n}_+$ with coefficients in $M \otimes \chi$ – the quantum Drinfeld-Sokolov reduction of M introduced in [4].

The analogue of the global Whittaker model should be a semi-infinite localization functor assigning a $\mathcal{D}$–module on each $S^n X$ to a $\widehat{\mathfrak{g}}(\mathbb{A})$–module $\otimes'_{x \in X} M_x$. In the case of $\widehat{\mathfrak{sl}}_2$ the fiber of this $\mathcal{D}$–module at the divisor $\sum_{x \in X} n_x[x] \in \mathrm{Div}_+$ should be the tensor product over all $x \in X$ of the semi-infinite cohomologies of $L\mathfrak{n}_{+,x} = \mathfrak{n}_+ \otimes \mathbb{C}((t_x))$ with coefficients in $M_x \otimes \chi_x$, where χ_x is a character of $L\mathfrak{n}_{+,x}$ vanishing at the origin up to order n_x.

We can apply this functor to $\otimes_{x \in X} V^{\rho_x}$, where ρ is a projective connection on X. In this case one can show that the fiber of the corresponding $\mathcal{D}$–module on $S^n X$ at the divisor $\sum_{x \in X} n_x[x]$ is isomorphic to $\otimes_x S^{n_x}(\mathbb{C}^2)$. Therefore we believe that this $\mathcal{D}$–module is isomorphic to the nth symmetric power of the $\mathcal{D}$–module on X defined by ρ.

6.5. **Genus zero case.** Let us consider the genus zero case allowing parabolic structures at the points $z_1, \ldots, z_N$ and ∞; Drinfeld has generalized his construction in [57] to allow for parabolic structures.

Put $n = N - 1$; Jac_n is just one point, and we have a pair of dual projective spaces: $\mathbb{P}^\vee = \mathbb{P}\,\mathrm{Ext}(\mathcal{O}(z_1 + \ldots + z_N + \infty), \mathcal{O})$ and $\mathbb{P} = \mathbb{P}H^0(\mathbb{P}^1, \Omega(z_1 + \ldots + z_N + \infty))$. The latter is isomorphic to $S^{N-1}\mathbb{P}^1$: if X_i's are the residues of one forms from $\mathbb{P}$ and $y_1, \ldots, y_{N-1}$ are natural coordinates on $S^{N-1}\mathbb{P}^1$, the isomorphism is given by

$$(6.5) \qquad \sum_{i=1}^N \frac{X_i}{t - z_i} dt = r \frac{\prod_{j=1}^{N-1}(t - y_j)}{\prod_{i=1}^N(t - z_i)} dt,$$

i.e. by the passage from a one-form to its zeroes.

There is a natural map $\mathbb{P}^\vee \to \mathcal{M}^{(N)}$. Namely, an element of $\mathbb{P}^\vee$ defines an extension $0 \to \mathcal{O} \to \mathcal{L}_2 \to \mathcal{O}(z_1 + \ldots + z_N + \infty) \to 0$. Then the canonical map $\mathcal{O} \to \mathcal{O}(z_1 + \ldots + z_N + \infty)$ provides us with an extension $\mathcal{L}'_2$ of $\mathcal{O}$ by $\mathcal{O}$ with flags at $z_1, \ldots, z_N, \infty$. The fiber of the map $\mathbb{P}^\vee \to \mathcal{M}^{(N)}$ is isomorphic to $\mathbb{P}^1$ without $N + 1$ points – this is the set of sections of $\mathcal{L}'_2$, which miss all flags, up to a scalar. There is a nice geometric approximation of this map, due to M. Thaddeus.

The $\mathcal{D}$–module $\Delta(\lambda_i, \mu_i)$ describes the Gaudin equations $H_i\Psi(X_1, \ldots, X_N) = \mu_i\Psi(X_1, \ldots, X_N), i = 1, \ldots, N$, which are equivalent to the equations

$$(6.6) \qquad S(t)\Psi = \left(\sum_{i=1}^N \frac{\lambda_i(\lambda_i + 2)/4}{(t - z_i)^2} + \sum_{i=1}^N \frac{\mu_i}{t - z_i} \right) \Psi, \quad t \in \mathbb{C} \backslash \{z_1, \ldots, z_N\},$$

where $S(t)$ is given by formula (5.7) and the action of the ith copy of $\mathfrak{sl}_2$ is given by formula (5.4). The pull-back of this system to $\mathbb{P}^\vee$ is the same system in which x_i's should be considered as the homogeneous coordinates on $\mathbb{P}^\vee$ dual to the coordinates X_i on $\mathbb{P}$.

The Radon transform is equivalent to the formal Fourier transform, i.e. substituting $x_i \to -\partial/\partial X_i$, and $\partial/\partial x_i \to X_i$ in formula (5.4), so that the action of the ith copy of

$\mathfrak{sl}_2$ becomes

$$(6.7) \qquad e^{(i)} = -X_i^2 \frac{\partial}{\partial X_i} - (\lambda_i + 2)X_i, \quad h^{(i)} = -2X_i \frac{\partial}{\partial X_i} - (\lambda_i + 2), \quad f^{(i)} = X_i.$$

We now have to rewrite the system (6.6) in the new coordinates y_j's using formula (6.5).

6.6. **Separation of variables.** This has been done by E. Sklyanin, using a clever trick [12]. Observe that $S(t) = f(t)e(t) + \frac{1}{4}h(t)^2 - \frac{1}{2}\partial_t h(t)$, where we use notation (5.5). After the Fourier transform, $f(t) = \sum_{i=1}^N X_i/(t - z_i)$, and according to formula (6.5), $f(t)|_{t=y_j} = 0$.

On the other hand, we have by (6.7):

$$\frac{1}{2}h(t) = -\sum_{i=1}^N \left(\frac{X_i \partial/\partial X_i}{t - z_i} - \frac{\lambda_i + 2}{t - z_i} \right).$$

But we find from (6.5):

$$\frac{\partial}{\partial y_j} = \sum_{i=1}^N \frac{X_i \partial/\partial X_i}{y_j - z_i}.$$

Hence

$$\frac{1}{2}h(t)|_{t=y_j} = -\nabla_{y_j} \equiv -\frac{\partial}{\partial y_j} + \frac{1}{2}\sum_{i=1}^N \frac{\lambda_i + 2}{y_j - z_i}.$$

Substituting $t = y_j$ from the left into $S(t)$ and using formulas above, we obtain [12]: $S(t)|_{t=y_j} = \nabla_{y_j}^2$.

Thus, substituting $t = y_j$ from the left into (6.6) we obtain the separated equations:

$$(6.8) \qquad \left(\nabla_{y_j}^2 - \sum_{i=1}^N \frac{\mu_i}{y_j - z_i} - \sum_{i=1}^N \frac{\lambda_i(\lambda_i + 2)/4}{(y_j - z_i)^2} \right) \Psi = 0, \quad j = 1, \dots, N - 1,$$

on symmetric polynomials $\Psi(y_1, \dots, y_{N-1})$. By a standard interpolation argument, we see that the systems (6.6) and (6.8) are equivalent.

Note that each of the equations (6.8) has the form $(\nabla_t^2 - q(t))\Psi = 0$, where $\partial_t^2 - q(t)$ is the projective connection (5.1) from which the original $\mathcal{D}$–module $\Delta(\lambda_i, \mu_i)$ was obtained. The replacement of ∂_t by ∇_t just means twisting by a rank one local system, which does not change our PGL_2–oper.

The perverse sheaf, which corresponds to the $\mathcal{D}$–module on $S^{N-1}\mathbb{P}^1$ defined by the system (6.8), is isomorphic to $E^{(N-1)}$, where E is the sheaf of local solutions of the projective connection $\partial_t^2 - q(t)$. Thus, the equivalence of Drinfeld's first and Beilinson-Drinfeld second constructions amounts in genus zero to separation of variables in the Gaudin equations. We will discuss this construction and its generalizations in detail elsewhere.

The separation of variables has been found by Sklyanin in an attempt to find an alternative to the conventional Bethe ansatz [13, 12, 14]. Sklyanin's approach goes back to the classical Hamilton-Jacobi theory of separation of variables in hamiltonian systems. In the theory of non-linear equations, which are integrated by the inverse

scattering method, the separating variables are usually the zeroes of the Baker function. Sklyanin's idea was that the function $f(t)$ above should be viewed as an analogue of the Baker function, and hence the equations should separate with respect to its zeroes. One of the advantages of this approach is that one does not need the "pseudo-vacuum" $|0\rangle$ for constructing eigenvectors, so representations without highest weight can also be considered. It also gives the most straightforward answer to the completeness problem.

Indeed, the Verma module M_{χ_i} over $\mathfrak{sl}_2$ can be realized in the space of polynomials $\mathbb{C}[X_i]$ with the action given by formula (6.7) with $\chi_i = -\lambda_i - 2$. A solution of the system (6.8) is a product of solutions with respect to each of the variables: $\Psi(y_1, \dots, y_{N-1}) = \prod_{i=1}^{N-1} \psi(y_i)$. Thus, a polynomial solution of (6.8) should be of the form $\Psi(y_1, \dots, y_{N-1}) = \prod_{i=1}^{N-1} \prod_{j=1}^{m} (y_i - w_j)$. Using formula (6.5) we can pass to the X_i–coordinates, and obtain an eigenvector of the Gaudin hamiltonians in $\otimes_{i=1}^{N} M_{\chi_i}$. It coincides with the vector $|w_1, \dots, w_m\rangle$ given by (5.6) up to a scalar multiple. Since the Gaudin equations on $\otimes_{i=1}^{N} M_{\chi_i}$ are equivalent to equations (6.8), we obtain completeness of Bethe ansatz for $\otimes_{i=1}^{N} M_{\chi_i}$. For other applications of the separation of variables, cf. [58].

There should exist a similar separation of variables for the generalized (spin) Calogero-Sutherland models defined on the moduli space of rank two bundles over the torus with parabolic structures.

6.7. **Generalization to higher rank.** G. Laumon [11] suggested a geometric construction of Langlands correspondence in the case of GL_n generalizing V. Drinfeld's first construction for GL_2 and P. Deligne's construction for GL_1. The idea is to connect the moduli space of rank $r > 1$ bundles with a section and the moduli space of rank $r + 1$ bundles with a section, by a pair of dual projective fibrations of the kind given by the diagram above. This way one continues the diagram further to the left and ultimately reaches the moduli space of rank n bundles. In order to construct a sheaf on the latter, one has to apply a chain of Radon transforms to the sheaf $E^{(m)}$ on $S^m X$, where E is now a rank n local system on X. The difficulty is to prove analogues of the Deligne vanishing theorem (cf. § 6.2), which are necessary to insure that the Radon transforms do not spoil irreducibility and perversity of sheaves, cf. [11].

It would be interesting to compare Laumon's construction, more precisely, its generalization to the bundles with parabolic structures, to the Beilinson-Drinfeld construction in genus zero. This should provide a separation of variables for Gaudin models of higher rank. For instance, in the case of SL_3, the Gaudin system is a system of differential equations in $3N$ variables, corresponding to an $\mathfrak{sl}_3$–oper ρ, i.e. a third order differential operator (5.13). We first have to apply a formal Fourier transform with respect to $2N$ of them, and make a change of variables analogous to (6.5). Then we have to apply another Fourier transform with respect to N variables, and apply another change of variables. This should give us a system of identical equations of the form $\rho \cdot \Psi = 0$. On the other hand, for the classical (and partially for quantum) Gaudin models associated to SL_n, a different scheme of separation of variables has been suggested in [59, 60], and in [13] a separation of variables has been given for the Calogero-Sutherland model corresponding to SL_3 and genus one. So there may exist another generalization of Drinfeld's first construction to groups of higher rank.

6.8. **Quantization.** As we already mentioned in the introduction, Sklyanin has found a quantum deformation of the separation of variables of the Gaudin equations [14], in which the role of the differential equations of second order (6.8) is played by q–difference equations of second order. This suggests that elements of the spectrum of the center $Z_q(\widehat{\mathfrak{g}})$ of a completion of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ at the critical level should be viewed as q–difference operators.

Explicit computation of the q–deformation of the Miura transformation made in [18] shows that it is indeed so. For example, $Z_q(\widehat{\mathfrak{sl}}_2)$ is generated by the Fourier components of a generating function $\ell(z)$, which is a q–deformation of the Sugawara operator $S(z)$. The q–deformation of the Miura transformation (3.4) reads:

$$(6.9) \qquad\qquad \ell(z) \longrightarrow \Lambda(qz) + \Lambda(zq^{-1})^{-1},$$

where $\Lambda(z)$ is a q–deformation of $\chi(z)$. According to this formula, $\ell(z)$ is a "quantum trace" of the conjugacy class of $\mathrm{diag}[\Lambda(z), \Lambda(z)^{-1}]$. Therefore elements of the spectrum of $Z_q(\widehat{\mathfrak{sl}}_2)$ look very similar to the local Langlands parameters described in § 4.2.

Formula (6.9) can be rewritten as follows: $\mathcal{D}_q^2 - \ell(z)\mathcal{D}_q + 1 = (\mathcal{D}_q - \Lambda(zq))(\mathcal{D}_q - \Lambda(zq)^{-1})$, where $[\mathcal{D}_q \cdot f](z) = f(zq^{-2})$. This is a q–analogue of (3.4). If $Q(z)$ is a solution of the q–difference equation $Q(zq) = \Lambda(z)Q(zq^{-1})$, then $(\mathcal{D}_q + \mathcal{D}_q^{-1} - \ell(z))Q(z) = 0$. This formula, which is a q–analogue of (6.8), coincides with the equation which appears in quantum separation of variables [14]. It plays a prominent role in statistical mechanics, cf. [62].

Quantum affine algebras may shed new light on the isomorphism between the center at the critical level and a $\mathcal{W}$–algebra. Indeed, the R–matrix of $U_q(\widehat{\mathfrak{g}})$ corresponding to a finite-dimensional representation coincides with the R–matrix of the quantum Toda system associated to $\widehat{\mathfrak{g}}$ [61]. The transfer-matrices of the Toda system are the integrals of motion of the latter, and are closely related to the central elements of $U_q(\widehat{\mathfrak{g}})$ at the critical level constructed in [15], cf. [18]. On the other hand, the classical $\mathcal{W}$–algebra $\mathcal{W}(\mathfrak{g}^L)$ is just the algebra of integrals of motion of the classical Toda system associated to $\mathfrak{g}^L$, and $\mathcal{W}(\mathfrak{g}^L)$ contains integrals of motion of the Toda system associated to $\widehat{\mathfrak{g}^L}$, cf. [63, 36]. However, the appearance of the Langlands dual Lie algebra here still remains a mystery.

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